Real Analysis Qualifying Exam Prep

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Contents

1 Real Analysis

1.1 Dominated Convergence

Theorem 1.1.1. *Let $(X, \mathscr{M}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of integrable functions with values in $\mathbb{C}$. Suppose for $g \geq 0$, g is integrable, $|f_n(x)| \leq g(x)$ for all x. Now suppose that $\lim_{n\to\infty} f_n(x) = f(x)$ for all x, then*

$$\int_X f_n(x)\, d\mu = \int_X f\, d\mu.$$

Dominated Convergence Example Problems (2)

Example 1.1.2 (Spring 2024, Problem 1)**.** Let $\tau_h f(x) = f(x-h)$ be the translation. Find all $p \in [1, \infty]$ for which $f \in L^p(\mathbb{R})$ implies that $\lim_{h\to 0} |f - \tau_h f|_{L^p(\mathbb{R})} = 0$ (and justify your answer).

Solution. Suppose g is a continuous function with compact support. Then we know that for sufficiently small h we have $|\tau_h g - g|^p \leq 2^p |g|^p$ which is integrable since $g \in L^p$. Since $\lim_{h\to 0} \tau_h g = g$ pointwise, by the dominated convergence theorem we have that

$$\lim_{h\to 0} \left(\int |\tau_h g - g|^p \right)^{1/p} = \left(\int \lim_{h\to 0} |\tau_h g - g|^p \right)^{1/p} = 0.$$

So for all continuous functions with compact support g and $\varepsilon > 0$ there exists h such that $\|\tau_h g - g\| < \varepsilon$. If f is not a continuous function with compact support, then there exists a continuous function g with compact support such that $\|f - g\|_p < \varepsilon$. Then by the translation invariance of the Lebesgue measure we have

$$\|\tau_h f - f\|_p \leq \|\tau_h f - \tau_h g\|_p + \|\tau_h g - g\|_p + \|g - f\|_p = 2\|f - g\|_p + \|\tau_h g - g\|_p < 3\varepsilon.$$

Thus we also have that

$$\lim_{h\to 0} \left(\int |\tau_h f - f|^p \right)^{1/p} = 0.$$

Example 1.1.3 (Spring 2023, Problem 3)**.** Let $f \in L^1([0,1])$ and let $f_n \to f$ pointwise a.e. in $[0,1]$. Show that

$$\lim_{n\to\infty} \int_0^1 ||f_n - f| - |f_n| + |f||\, dx = 0.$$

Solution. Note that using both the triangle inequality and reverse triangle inequality

$$||f_n - f| - |f_n| + |f|| \leq ||f_n - f| - |f_n|| + |f| \leq 2|f|.$$

Since $f \in L^1([0,1])$, by the above inequality we find that our integrand is suitably dominated. Since $f_n \to f$ we know that $|f_n - f| \to 0$ and $|f_n| \to |f|$. Thus $||f_n - f| - |f_n| + |f|| \to 0$. So by the dominated convergence theorem

$$\lim_{n\to\infty} \int_0^1 ||f_n - f| - |f_n| + |f||\, dx = \int_0^1 \lim_{n\to\infty} ||f_n - f| - |f_n| + |f||\, dx = \int_0^1 0\, dx = 0.$$

1.2 Monotone Convergence Theorem

Theorem 1.2.1. *Let $\{f_n\}$ be a pointwise non-decreasing series of measurable functions, then we have that*

$$\lim_{n\to\infty}\int f_n = \int \lim_{n\to\infty} f_n.$$

Theorem 1.2.2. *Let $\{f_n\}$ be a pointwise non-increasing series of measurable functions, then if $f_1 \in L^1$ we have*

$$\lim_{n\to\infty}\int f_n = \int \lim_{n\to\infty} f_n.$$

Monotone Convergence Theorem Example Problems (1)

Example 1.2.3 (Fall 2022, Problem 5)**.** Prove that if f is Lebesgue integrable on A, then for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_B |f(x)|\,dx < \varepsilon$ for all $B \subseteq A$ with $\mu(B) < \delta$.

Solution. Let $f_n = \min(|f|, n)$ and note that $|f| - |f_n| \to 0$ pointwise monotonically decreasing. Since $f \in L^1$ and $|f_1| \leq |f|$ we have that $|f| - |f_1| \in L^1$ also. Thus by the monotone convergence theorem we have that

$$\lim_{n\to\infty}\int_A |f| - |f_n|\,dx = \int_A \lim_{n\to\infty} |f| - |f_n|\,dx = \int_A 0\,dx = 0.$$

Thus we can choose n sufficiently large such that

$$\int_A |f| - |f_n|\,dx < \frac{\varepsilon}{2}.$$

Now if we choose $\delta = \varepsilon/(2n)$ then we have that

$$\int_B |f|\,dx = \int_B |f| - |f_n|\,dx + \int_B |f_n|\,dx < \int_A |f| - |f_n|\,dx + n\,\mu(B) < \frac{\varepsilon}{2} + n\left(\frac{\varepsilon}{2n}\right) = \varepsilon.$$

1.3 Modes of Convergence

Modes of Convergence Example Problems (2)

Example 1.3.1 (Fall 2023, Problem 1)**.** Let $f_n \to f$ in measure on the metric space (X, μ). Show there exists a subsequence f_{n_k} such that $f_{n_k} \to f$ pointwise μ a.e.

Solution. Recall that since $f_n \to f$ in measure, for all $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

Thus there exists an non-decreasing sequence $\{n_k\}$ such that

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq k^{-1}\}) < 2^{-k}$$

for all $n \geq n_k$. Now let

$$E_k = \{x \in X : |f_{n_k}(x) - f(x)| \geq k^{-1}\} \qquad \text{and} \qquad F_k = \bigcup_{j=k}^{\infty} E_j.$$

Note $\mu(E_k) < 2^{-k}$ and $\mu(F_k) < 2^{1-k}$. Now let $F = \bigcap_{k\in\mathbb{N}} F_k$; then $\mu(F) \leq \mu(F_k) < 2^{1-k}$ for all k, thus $\mu(F) = 0$.

If $x \notin F$ then we have that $x \notin F_k$ for some k and so $x \notin E_j$ for all $j \geq k$. Thus

$$x \in \{x \in X : \left|f_{n_j}(x) - f(x)\right| < j^{-1}\} \qquad \text{for all} \qquad j \geq k.$$

This of course implies that $f_{n_k} \to f$ pointwise for $x \notin F$ which has measure zero.

Example 1.3.2 (Fall 2020, Problem 2)**.** Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of integrable functions with $\int_{\mathbb{R}} |f_n| \leq 1$. Assume there exists an f such that $f_n \to f$ in measure. Show that there exists a subsequence f_{n_k} such that $f_{n_k} \to f$ pointwise μ a.e. and that f is integrable.

Solution. The first part of the problem follows as the one directly before this (Fall 2023, Problem 1); thus, there exists a subsequence f_{n_k} such that $f_{n_k} \to f$ pointwise a.e. Since there exists a subsequence $f_{n_k} \to f$ we know that $|f| \leq \liminf_{n\to\infty} |f_n|$. So by Fatou's lemma we have

$$\int |f| \leq \int \liminf_{n\to\infty} |f_n| \leq \liminf_{n\to\infty} \int |f_n| \leq 1.$$

Thus f is integrable.

1.4 Holder's Inequality

Theorem 1.4.1. *We have that*

$$\int |fg| \le \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}$$

where $p^{-1} + q^{-1} = 1$.

Holder's Inequality Example Problems (2)

Example 1.4.2 (Spring 2021, Problem 3)**.** Let $1 \le p < \infty$ and show that

$$\left(\int_Y \left(\int_X F(x,y)\, d\mu(x)\right)^p d\nu(y)\right)^{1/p} \le \int_X \left(\int_Y F(x,y)^p\, d\nu(y)\right)^{1/p} d\mu(x).$$

Solution. ???

Example 1.4.3 (Fall 2023, Problem 2)**.** Let $a_n \ge 0$ for $n \in \mathbb{N}$ and $0 < p < q$.

- Prove that
$$\left(\sum_{n\in\mathbb{N}} a_n^q\right)^{1/q} \le \left(\sum_{n\in\mathbb{N}} a_n^p\right)^{1/p}.$$
- Prove that for every $N \in \mathbb{N}$ we have
$$\left(\sum_{n=1}^N a_n^p\right)^{1/p} \le N^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{n=1}^N a_n^q\right)^{1/q}.$$

Solution. Let $b_n = a_n/\|a\|_p$. Then $b_n \le 1$ for all n and $\|b\|_p = 1$. Thus

$$\|b\|_q \le \left(\sum_{n\in\mathbb{N}} b_n^p\right)^{1/q} = \|b\|_p^{p/q} = 1^{p/q} = 1.$$

Thus we have

$$\|a\|_q = \Big\| \|a\|_p \cdot b \Big\|_q = \|a\|_p \|b\|_q \le \|a\|_p.$$

Now note that $(q/p)^{-1} + (q/(q-p))^{-1} = 1$. So by Holder's inequality we have

$$\sum_{n=1}^N a_n^p \le \left(\sum_{n=1}^N a_n^q\right)^{p/q} \left(\sum_{n=1}^N 1^{q/(q-p)}\right)^{(q-p)/q} = N^{(q-p)/q} \left(\sum_{n=1}^N a_n^q\right)^{p/q}.$$

Taking the p-th root gives the desired result.

1.5 Egorov's Theorem

Theorem 1.5.1. *Let $f_n \to f$ pointwise on X. For all $\varepsilon > 0$ there exists $E \subseteq X$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on $X \setminus E$.*

Egorov's Theorem Example Problem (1)

Example 1.5.2 (Spring 2024, Problem 3)**.** Let f_n, g_n be sequences of functions in $L^2([0,1])$. For $f \in L^2([0,1])$ and $g : [0,1] \to \mathbb{R}$ measurable we have $f_n \to f$ in $\|\cdot\|_2$ and $g_n \to g$ pointwise a.e. Also assume that $\|g_n\|_2 \leq 1$. Show that

$$\int f_n g_n \to \int fg.$$

Solution. First note that

$$\int |f_n g_n - fg| \leq \int |f_n g_n - f g_n| + \int |f g_n - fg|. \tag{1}$$

Now note that by Cauchy-Schwarz and $\|g_n\|_2 \leq 1$ we have

$$\int |f_n g_n - fg| \leq \left(\int |g_n|^2\right)^{1/2} \left(\int |f_n - f|^2\right)^{1/2} \leq \left(\int |f_n - f|^2\right)^{1/2}.$$

Since $f_n \to f$ in $\|\cdot\|_2$ we have that

$$\lim_{n\to\infty} \int |f_n g_n - fg| \leq \lim_{n\to\infty} \left(\int |f_n - f|^2\right)^{1/2} = 0. \tag{2}$$

Now note by Cauchy-Schwarz and $f \in L^2([0,1])$, there exists C such that

$$\int |f g_n - fg| \leq \left(\int |f|^2\right)^{1/2} \left(\int |g_n - g|^2\right)^{1/2} = C\left(\int |g_n - g|^2\right)^{1/2}.$$

Now by Egorov's theorem we know that for all $\varepsilon > 0$ there exists a set $E \subseteq [0,1]$ with $\mu(E) < \varepsilon$ such that $g_n \to g$ uniformly on $[0,1] \setminus E$. Thus we have

$$\lim_{n\to\infty} |f g_n - fg| \leq \lim_{n\to\infty} C\left(\int |g_n - g|^2\right)^{1/2} = C\left(\lim_{n\to\infty}\int_E |g_n - g|^2\right)^{1/2} + C\left(\int_{[0,1]\setminus E} \lim_{n\to\infty} |g_n - g|^2\right)^{1/2}.$$

Since $g_n \to g$ pointwise, the integral over $[0,1] \setminus E$ vanishes. Thus

$$\lim_{n\to\infty} |f g_n - fg| \leq C\left(\int_E |g_n - g|^2\right)^{1/2}.$$

Recall that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $\int_E |g_n - g|^2 < \varepsilon$. Thus $\lim_{n\to\infty} |f g_n - fg| = 0$. Putting this together with (1) and (2), we have that

$$\lim_{n\to\infty} \int |f_n g_n - fg| \leq \lim_{n\to\infty} \int |f_n g_n - f g_n| + \lim_{n\to\infty} \int |f g_n - fg| = 0.$$

1.6 Fatou's Lemma

Lemma 1.6.1. *Let f_n be a sequence of non-negative measurable functions. Then we have that*

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.$$

Fatou's Lemma Example Problems (1)

Example 1.6.2 (Fall 2022, Problem 1)**.** Let $f : [0,1] \to [0,1]$ be a continuous non-decreasing function with $f(0) = 0$ and $f(1) = 1$. For this function, f' exists almost everywhere on $[0,1]$. Let dx denote the Lebesgue measure.

- Use Fatou's Lemma to show that $\int_0^1 f'(x)\,dx \leq 1$.
- Provide an example of a function f such that $\int_0^1 f'(x)\,dx < 1$ strictly.

Solution. Let $f_n(x) = n\,(f(x+1/n) - f(x))$ on $x \in [0, 1-1/n)$ and 0 on $x \in [1-1/n, 1]$. Now note that

$$\int_0^1 f_n(x)\,dx = n\int_{1/n}^1 f(x)\,dx - n\int_0^{1-1/n} f(x)\,dx = n\int_{1-1/n}^1 f(x) - n\int_0^{1/n} f(x)\,dx \leq n\left(\frac{f(1)}{n} - \frac{f(0)}{n}\right) = 1.$$

Note that $f_n \to f'$ pointwise on $[0,1)$; additionally, $f_n(x)$ is non-negative since f is non-decreasing, by Fatou's lemma

$$\int_0^1 f'(x)\,dx = \int_0^1 \liminf_{n\to\infty} f_n(x)\,dx \leq \liminf_{n\to\infty} \int_0^1 f_n(x)\,dx \leq 1.$$

An example of a function in which the strict inequality holds is the Cantor function C. This function maps $[0,1] \to [0,1]$, is continuous and non-decreasing, satisfies $C(0) = 0$ and $C(1) = 1$ and is differentiable almost everywhere; however, because it has zero derivative almost everywhere we have

$$\int_0^1 C'(x)\,dx = 0 < 1.$$

Example 1.6.3 (Spring 2021, Problem 4)**.** Let $(X, \mathcal{B}(X), \mu)$ be a measure space. If $f_n, g_n, f, g \in L^1(X,\mu)$ and

- $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$ a.e.
- $|f_n| \leq g_n$ for all $n \in \mathbb{N}$
- $\lim_{n\to\infty} \int_X g_n\,dx = \int_X g\,dx$.

Then we have that

$$\lim_{n\to\infty} \int_X f_n\,dx = \int f\,dx$$

Solution. Note that

$$\int g - \int f = \int (g-f) = \int \liminf_{n\to\infty} (g_n - f_n) \leq \liminf_{n\to\infty} \int (g_n - f_n) = \int g - \limsup_{n\to\infty} \int f_n.$$

Additionally,

$$\int g + \int f = \int (g+f) = \int \liminf_{n\to\infty} (g_n + f_n) \leq \liminf_{n\to\infty} \int (g_n + f_n) = \int g + \liminf_{n\to\infty} \int f_n.$$

Thus $\limsup_{n\to\infty} \int f_n \leq \int f \leq \liminf_{n\to\infty} \int f_n$ and the desired result follows.

1.7 Set Theory

Set Theory Example Problems (4)

Example 1.7.1 (Fall 2020, Problem 1)**.** Let E_k be a sequence of Lebesgue measurable subsets of $\mathbb{R}$. Let

$$E = \{x \in \mathbb{R} : x \in E_k \text{ for infinitely many } k\}.$$

1. Show that E is Lebesgue measurable.
2. Show that if $\sum |E_k| < \infty$ then $|E| = 0$.
3. Assume instead only that $\lim_{k\to\infty} |E_k| = 0$. Must $|E| = 0$?

Solution. Suppose that

$$E = \bigcap_{j\in\mathbb{N}} \bigcup_{k\geq j} E_k,$$

we will show that this is indeed the case. Note that if $x \in \mathbb{R}$ is in finitely many E_k, then there exists K such that $x \notin E_k$ for all $k \geq K$. Thus $x \notin \bigcup_{k\geq K} E_k$ and so $x \notin E$. If $x \in \mathbb{R}$ is in infinitely many E_k, then $x \in \bigcup_{k\geq j} E_k$ for all j; thus, $x \in E$. We know that Lebesgue measurable sets are closed under countable union and intersections, thus E is Lebesgue measurable.

Let $f(k, x) = 1$ if $x \in E_k$ and 0 otherwise. Now let $g(x) = \sum_{k\in\mathbb{N}} f(k, x)$. Now note that

$$\sum_{k\in\mathbb{N}} \int_{\mathbb{R}} f(k, x)\, dx = \sum_{k\in\mathbb{N}} |E_k| < \infty.$$

Thus since $f(k, x)$ is strictly positive, by Fubini we have

$$\int_{\mathbb{R}} g(x)\, dx = \int_{\mathbb{R}} \sum_{k\in\mathbb{N}} f(k, x)\, dx = \sum_{k\in\mathbb{N}} \int_{\mathbb{R}} f(k, x)\, dx = \sum_{k\in\mathbb{N}} |E_k| < \infty.$$

Now if we suppose that $|E| > 0$ then by observing that $x \notin E$ if and only if $g(x) < \infty$, we arrive at a contradiction. Thus it must be the case that $|E| = 0$.

No, it is not necessarily true that $|E| = 0$ if $\lim_{k\to\infty} |E_k| = 0$. Consider the typewriter sequence.

Example 1.7.2 (Spring 2021, Problem 5)**.** Assume that $E \subseteq \mathbb{R}$ is Lebesgue measurable and $0 < m(E) < \infty$.

- Show that if E is bounded and $m(E) = p > 0$, then for each $q \in (0, p)$ there exists a measurable $B \subseteq E$ with measure q.
- Prove that for any $0 < \alpha < 1$ there exists an open interval such that

$$\alpha\, m(I) \leq m(E \cap I).$$

Solution. Let $f(x) = m(E \cap (-\infty, x))$, note that this function is continuous. Since E is bounded we know there exists M such that $|x| > M$ implies $x \notin E$. Thus $f(-M) = 0$ and $f(M) = m(E) = p$. By the intermediate value theorem there exists an x with $|x| \leq M$ such that $f(x) = q$ for all $q \in (0, p)$.

By the Lebesgue density theorem we know there exists a point $x \in E$ such that

$$\lim_{r\to 0^+} \frac{m(E \cap (x - r, x + r))}{m((x - r, x + r))} = 1.$$

Thus for all $0 < \alpha < 1$ there exists sufficiently small r such that

$$\frac{m(E \cap (x - r, x + r))}{m((x - r, x + r))} > \alpha.$$

Rearranging this inequality gives the desired result.

Example 1.7.3 (Spring 2022, Problem 3/Fall 2024, Problem 2)**.** Let m denote the Lebesgue measure on $\mathbb{R}$ and let $I = [0,1]$. Decide whether the sets below are closed in $L^1(I,m)$ where

$$A = \left\{ f \in L^1(I,m) : \int_I |f(x)|^2 \, dm \geq 1 \right\} \qquad \text{and} \qquad B = \left\{ f \in L^1(I,m) : \int_I |f(x)|^2 \, dm \leq 1 \right\}.$$

Solution. Consider the sequence of functions

$$f_n(x) = \begin{cases} \sqrt{n} & 0 \leq x < 1/n \\ 0 & \text{otherwise} \end{cases}.$$

Note that $f_n \to f = 0$ in $L^1(I,m)$. We have that

$$\int_I |f_n(x)|^2 \, dm = 1 \qquad \text{but} \qquad \int_I |f(x)|^2 \, dm = 0.$$

Thus A is not closed since $f_n \to f$ in $L^1(I,m)$ with all $f_n \in L^1(I,m)$ but $f \notin L^1(I,m)$.

We will now prove that B is closed. First note that convergence of $f_n \to f$ in $L^1(I,m)$ implies that $f_n^2 \to f^2$ in measure. By the definition of $\liminf$ there exists a subsequence $f_{n_k}^2$ such that

$$\liminf_{k\to\infty} \int_I f_{n_k}^2 \, dm = \liminf_{n\to\infty} \int_I f_n^2 \, dm.$$

By Theorem 2.30 in Folland, we know there exists $f_{n_{k_j}}^2$ which converges almost everywhere to f^2. Thus by Fatou's lemma we have

$$\int_I f^2 \, dm = \int_I \liminf_{j\to\infty} f_{n_{k_j}}^2 \, dm \leq \liminf_{j\to\infty} \int_I f_{n_{k_j}}^2 = \liminf_{n\to\infty} \int_I f_n^2 \, dm \leq 1.$$

Thus B is closed in $L^1(I,m)$.

Example 1.7.4 (Fall 2024, Problem 4)**.** Let μ^* denote Lebesgue outer measure on $\mathbb{R}$. Let A and B be any two subsets of $\mathbb{R}$ that are seperated by a positive distance d. That is, if $a \in A$ and $b \in B$ then $|a - b| \geq d > 0$. Show that

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

Solution. Since A and B are separated by some non-zero distance, we know that $\overline{A} \cap B = \emptyset$ (the closure of A intersect B is empty). Additionally, since $\overline{A}$ is closed, we know it is Lebesgue measurable, which means it satisfies Caratheodory's criterion.

$$\begin{aligned} \mu^*(A \cup B) &= \mu^*((A \cup B) \cap \overline{A}) + \mu^*((A \cup B) \cap (\mathbb{R} \setminus \overline{A})) \\ &= \mu^*((A \cap \overline{A}) \cup (B \cap \overline{A})) + \mu^*((A \cap (\mathbb{R} \setminus \overline{A})) \cup (B \cap (\mathbb{R} \setminus \overline{A}))) \\ &= \mu^*(A \cup \emptyset) + \mu^*(\emptyset \cup B) \\ &= \mu^*(A) + \mu^*(B). \end{aligned}$$

1.8 Other

Other Example Problems (4)

Example 1.8.1 (Fall 2020, Problem 4)**.** Construct a non-decreasing function $f : (0,1) \to \mathbb{R}$ whose discontinuity set is exactly $\mathbb{Q} \cap (0,1)$ or prove that such a function does not exist.

Solution. We know that $\mathbb{Q} \cap (0,1)$ is countably infinite, thus there exists a bijection $g : \mathbb{N} \to \mathbb{Q} \cap (0,1)$. Now let

$$f : (0,1) \to \mathbb{R} \qquad \text{via} \qquad f(x) = \sum_{g(n) \le x} 2^{-n}.$$

Since the series $\sum 2^{-n}$ is absolutely convergent, we know that any rearrangement of the series converges as well. Thus, f is well defined on $(0,1)$ and it is non-decreasing. If $x \le y$ and $g(n) \le x$ then $g(n) \le y$; thus,

$$f(y) = f(x) + \sum_{x < g(n) \le y} 2^{-n} \ge f(x).$$

Additionally, it is clear that f is discontinuous on $\mathbb{Q} \cap (0,1)$.

Example 1.8.2 (Fall 2023, Problem 3)**.** Show that if $f(x,y) = ye^{-(1+x^2)\,y^2}$ then

$$\int_0^\infty \int_0^\infty f(x,y)\,dx\,dy = \int_0^\infty \int_0^\infty f(x,y)\,dy\,dx.$$

Use this to show that

$$\int_{\mathbb{R}} e^{-x^2} = \sqrt{\pi}.$$

Solution. Since $f(x,y)$ is strictly positive, we know that if either integral converges, then they are necessarily equal by Fubini. Note that

$$\int_0^\infty \int_0^\infty ye^{-(1+x^2)\,y^2}\,dy\,dx = \frac{1}{2}\int_0^\infty \int_0^\infty e^{-(1+x^2)\,u}\,du\,dx = \frac{1}{2}\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{4}.$$

Thus,

$$\int_0^\infty \int_0^\infty f(x,y)\,dx\,dy = \int_0^\infty \int_0^\infty f(x,y)\,dy\,dx = \frac{\pi}{4}.$$

Now note that

$$\frac{\pi}{4} = \int_0^\infty \int_0^\infty f(x,y)\,dx\,dy = \int_0^\infty ye^{-y^2} \int_0^\infty e^{-(xy)^2}\,dx\,dy = \int_0^\infty e^{-y^2} \int_0^\infty e^{-u^2}\,du\,dy = \left(\int_0^\infty e^{-x^2}\,dx\right)^2.$$

Thus since $\exp\!\left(-x^2\right)$ is symmetric in x we have

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-x^2}\,dx \qquad \Longrightarrow \qquad \int_{\mathbb{R}} e^{-x^2}\,dx = \sqrt{\pi}.$$

Example 1.8.3 (Spring 2024, Problem 2)**.** Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of measurable functions with $|f_n(x)| \leqslant 1$ for a.e. x. Let

$$g_n(x) = \int_0^x f_n(t)dt$$

Show that there exists an absolutely continuous g and a subsequence $n_k \to \infty$ such that $g_{n_k} \to g$ in $C([0,1])$.

Solution. Note that

$$|g_n(x)| = \left|\int_0^x f_n(t)\,dt\right| \le \int_0^x |f_n(t)|\,dt \le \int_0^x dt = x.$$

Thus $|g_n(x)| \le x$ uniformly in n. Similarly,

$$|g_n(y) - g_n(x)| = \left|\int_x^y f_n(t)\,dt\right| \le \int_x^y |f_n(t)|\,dt \le \int_x^y dt = y - x.$$

Thus $|g_n(y) - g_n(x)| \le |y - x|$ uniformly in n. Since the g_n are uniformly bounded and equicontinuous, by Arzela-Ascoli we know there exists a subsequence n_k such that $g_{n_k} \to g$ uniformly where g is in $C([0,1])$. Since g_n is the integral of an L^1 function, it is absolutely continuous. Since g is the uniform limit of absolutely continuous functions, it is absolutely continuous.

Example 1.8.4 (Fall 2024, Problem 1)**.** Let (X, ρ) be a metric space

- Suppose that (X, ρ) is separable. Prove that if $Y \subseteq X$ then (Y, ρ) is separable also.
- Suppose that (X, ρ) is compact. Let $\mathcal{F}$ be any set of real valued functions on X that is uniformly bounded and equicontinuous. Is the function $g(x) = \sup_{f \in \mathcal{F}} f(x)$ necessarily continuous?

Solution. Let $\rho(x, Y) = \inf\{\rho(x, y) : y \in Y\}$. Since (X, ρ) is separable there exists a countable dense subset $D \subseteq X$. Now for each $d \in D$ let us choose $e_{d,n} \in Y$ such that $\rho(d, e_{d,n}) < \rho(d, Y) + 1/n$. Now define

$$E = \bigcup_{d \in D} \bigcup_{n \in \mathbb{N}} e_{d,n}.$$

Note that E is a countable set since it is a countable union of countable sets.

Now let $\varepsilon > 0$ and $y \in Y$. Since D is a dense subset of X and $Y \subseteq X$, there exists $d \in D$ with $\rho(y, d) < \varepsilon$. Thus, $\rho(d, Y) < \varepsilon$ certainly. By construction of E we also know there exists $e \in E$ such that $\rho(d, e) < \rho(d, Y) + \varepsilon$. Thus,

$$\rho(y, e) \leq \rho(y, d) + \rho(d, e) < \rho(y, d) + \rho(d, Y) + \varepsilon < 3\varepsilon.$$

So E is a dense subset of Y.

Thus E is a countably dense subset of Y and so it follows that (Y, ρ) is separable.

Since $\mathcal{F}$ is equicontinuous, we know that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$ for all $f \in \mathcal{F}$. Thus we have

$$f(x) < f(y) + \varepsilon \leq g(y) + \varepsilon \qquad \text{and} \qquad f(y) < f(x) + \varepsilon \leq g(x) + \varepsilon.$$

So we have that

$$g(x) = \sup_{f \in \mathcal{F}} f(x) < \sup_{f \in \mathcal{F}} (g(y) + \varepsilon) = g(y) + \varepsilon \qquad \text{and} \qquad g(y) = \sup_{f \in \mathcal{F}} f(y) < \sup_{f \in \mathcal{F}} (g(x) + \varepsilon) = g(x) + \varepsilon.$$

Thus $|x - y| < \delta \implies |g(x) - g(y)| < \varepsilon$ as desired.

Example 1.8.5 (Fall 2021, Problem 1)**.** For a sequence of real numbers $\{a_n\}$, write down the definition of $\limsup_{n\to\infty} a_n$. Prove that for any sequence of Lebesgue measurable functions $\{f_n\}$ we have that $f = \limsup_{n\to\infty} f_n$ is Lebesgue measurable also.

Solution. Recall that $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \big(\sup_{m\geq n} a_n\big) = \inf_{n\geq 0} \big(\sup_{m\geq n} a_n\big)$.

Let $f_i = \inf_{n\in\mathbb{N}} f_n$ and $f_s = \sup_{n\in\mathbb{N}} f_n$. Note that

$$f_i^{-1}([-\infty, x)) = \bigcup_n f_n^{-1}([-\infty, x)) \qquad \text{and} \qquad f_s^{-1}((x, \infty]) = \bigcup_n f_n^{-1}((x, \infty]).$$

Since each f_n is measurable we know that $f_n^{-1}([-\infty, x))$ and $f_n^{-1}((x, \infty])$ are measurable. Since $f_i^{-1}([-\infty, x))$ and $f_s^{-1}((x, \infty])$ are a countable union of measurable sets, they are both measurable. Thus f_i and f_s are measurable functions, so the infinmum and supremum of a family of measurable functions is itself measurable.

Since $\limsup_{n\to\infty} f_n = \inf_{n\geq 0} \big(\sup_{m\geq n} f_n\big)$, it follows that $\limsup_{n\to\infty} f_n$ is measurable.