Oral Qualifying Exam Transcript

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My committee for this exam was: Alex Kontorovich (chair), Henryk Iwaniec, Matthew Young, and Mariusz Mirek. In this transcript they are denoted as AK, HI, MY, and MM respectively. I am denoted as PW. Questions asked may not be in order of which they appeared on the exam, but roughly follow the order in which they appear on my syllabus. If I am unsure of who asked what questions I will denote them using a slash (as an example I may write "HI/MY" to denote Henryk or Matthew). Of course there is room for me misremembering things as none of this was explicitly recorded, but this is as best as I can recall what happened.

MM: What can you tell me about Poisson summation.

PW: *I write down the Poisson summation formula.*

MM: For what types of functions is this valid?

PW: It is valid for Schwartz functions.

MM: What is an application of the Poisson summation formula?

PW: You can use it to derive the functional equation for θ, i.e. $\theta(1/t) = \sqrt{t}\,\theta(t)$.

HI: What is the θ function?

PW: It is $\theta(t) = \sum_n \exp\bigl(-\pi n^2 t\bigr)$.

HI/MY: What is the functional equation of the ζ function?

PW: *I write down the functional equation of the ζ function.*

MM: What does this tell you about the zeros of the zeta function?

PW: Rearranging we have that

$$\zeta(s) = \frac{\Gamma((1-s)/2)\,\zeta(1-s)\,\pi^{s/2}}{\Gamma(s/2)\,\pi^{(1-s)/2}}.$$

We know that if $\sigma < 0$ then $1 < 1-\sigma$, so $\zeta(1-s)$ is non-zero from the Euler product expansion. We know the other terms in the numerator are non-zero also, so any ζ zero comes from a pole in the denominator. Of course, $\Gamma(s/2)$ has poles at the negative even integers, and these give the trivial ζ zeros.

HI/MY/MM: Can you derive the functional equation for the ζ function?

PW: *I give the standard argument by looking at $\mathcal{M}((\theta(t)-1)/2)(s/2)$*

HI: Can you view θ from an automorphic form perspective?

PW: Hmm... I am unsure...

MY: Can you write down the functional equation of the Dirichlet L-function?

PW: *I write down the formula forgetting to conjugate χ in the $L(\overline{\chi}, 1-s)$ term and forgetting the root number, but I am able to correct my mistake before it is pointed out.*

MY: What happens if χ is a quadratic character?

PW: Hmm... I am unsure...

MY: What is the root number of a quadratic character?

PW: *I stare at the board blankly.*

AK/HI/MY: *At some point they start making me do explicit computations with quadratic characters on the board and it goes rather poorly.*

MY: So it turns out the root number for a quadratic character is always 1. This is something Henryk asked me on my qualifying exam; I was stumped as well, but ultimately was able to figure it out.

PW: Mmm... okay.

HI: Also... *Points out that the functional equation holds only for primitive χ, something I failed to mention.*

HI: How can you use the transcendence of π to prove there are infinitely many primes?

PW: How can I use the transcendence of π?

HI: Yeah.

PW: Hmm... I'm used to the argument using ζ as $s \to 1^+$...

PW: *After thinking for a bit* Oh I see what you are asking, I guess if you suppose there are finitely many primes then the Euler product of ζ tells you that $\zeta(2)$ is rational; but of course, $\zeta(2) = \pi^2/6$, which is not rational.

HI: Good. Yes.

AK/HI/MY: Can you state the prime number theorem?

PW: Yes, $\psi(x) = x + O(x \exp(-c\sqrt{\log x}))$.

HI/MM: If RH is true, what can you say about ψ?

PW: The error term should be like $x^{1/2+\varepsilon}$, so I guess $\sqrt{x} \log x$.

HI/MM: It should be $\sqrt{x} \log^2 x$, but yes, close enough.

MM: Why is this the case?

PW: Hmm... I guess this comes down to the $\log^2 |t|$ estimate on ζ'/ζ. In proving this estimate, we use Blaschke products to get something like the absolute value of the logarithmic derivative of ζ minus some sum over zeros of ζ is bounded above by $\log^2 |t|$ up to constant. So in the traditional zero free region, we are able to get the $\log^2 |t|$ estimate. If we have RH, we have the zero-free region up to $\sigma > 1/2$.

MM: What is the traditional zero-free region?

PW: It's $\sigma > 1 - C/\log |t|$ where C is some constant and $|t|$ is like greater than 3 or something.

HI: Okay let's move on, I wanted to ask this question, its something I ask everyone and everyone should know it. I told Matthew to ask this if my flight didn't return in time.

PW: ...

HI: What is the statement of Bomberi-Vinogradov?

PW: *I write down the statement of Bomberi-Vinogradov.*

HI: And what is significant about this?

PW: It gives an error term which is as good as RH on average.

AK: Okay lets move on, Henryk do you want to ask about some sieve content? Maybe the Λ^2-sieve or the large sieve?

HI: Hmm... I could ask about short character sums, but that is too hard/deep a topic...

AK: Maybe ask about Brun's theorem?

AK/HI: What is the central idea of the Λ^2-sieve?

PW: I guess the main idea is leveraging the inequality

$$\sum_{d|n} \mu(d) \leq \left(\sum_{d|n} \lambda(d) \right)^2$$

where λ is some sequence with $\lambda(1) = 1$.

HI: Yes, but why is this important.

PW: I guess it helps us get around some of the randomness of μ?

HI: That is a good answer, but not what I am looking for... Hmm...

PW: I mean it gives us a lot of flexibility in how we choose λ...

(We go back and forth waffling here, something about support is mentioned but I'm lost/unsure what to answer. Eventually I write...)

PW: I mean, I guess let me just write it all out. We eventually get the estimate

$$S(\mathcal{A}, P_z) = \sum_{(n,P_z)=1} a_n \leq \frac{x}{H} + (DH)^2.$$

HI: And what is D here?

PW: That is some arbitrary level of support for λ.

HI: Okay that is what I wanted.

AK: And how do we choose our λ?

PW: We choose them to optimize our estimate in a suitably convenient way using Lagrange multipliers.

AK: Okay, look at your $S(\mathcal{A}, P_z)$ on the board. When you apply the inequality what do you get.

PW: You get some double sum over $d, d' \mid P_z$ of $\lambda(d)\,\lambda(d')$ —

AK: And what degree is that term?

PW: A quadratic.

AK: Yes. We choose λ by optimizing some quadratic w.r.t a linear constraint.

PW: Sure. Okay.

AK: Okay, do we want to move onto modular forms content? Were running a little slow on time.

AK/MY: Okay, recall the proof of the ζ functional equation you did earlier? Can you do something similar for modular forms?

PW: Yeah, of course. Instead of looking at $\mathcal{M}((\theta(t)-1)/2)(s/2)$ we can look at $\mathcal{M}(f(it))(s)$ where $f \in S_k$ and play a similar game of splitting up the integral into two pieces and flipping the integral from 0 to 1 to be from 1 to ∞. Though I guess this is different in the general level case where you have to do extra work since the appropriate Fricke involution doesn't lie in $\Gamma_0(N)$.

AK/HI: What matrix does this flip correspond to?

PW: In the level 1 case or the level N case? Or, I guess I can just write both. In the level 1 case it corresponds to $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \Gamma$ and for level N we use the Fricke involution which corresponds to $\omega = \left(\begin{smallmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{smallmatrix}\right)$ which is not in $\Gamma_0(N)$ generally, but this is what the work of Hecke and Atkin-Lehner solves, so we know that $Wf = \pm f$ where Wf is slashing by ω. I guess this only works for Hecke eigenforms in the general level case.

HI: And who did this work?

PW: I guess Hecke, Atkin-Lehner, and Fricke.

HI: And what region are you integrating over?

PW: Do you mean like the bounds of integration on the integral, or where we can interchange the sum and the integral?

HI: Yes, where you can interchange the sum and integral.

PW: I guess this is valid on $1 + k/2 < \sigma$ by the Hecke bound.

AK/HI/MY: And what is the Hecke bound?

PW: It is $|a_n| \ll n^{k/2}$.

AK/HI/MY: Can you prove it?

PW: Yeah! So basically, we look at the sum

$$\sum_{n \leq N} |a_n|^2 e^{-4\pi ny} = \sum_{n \leq N} (a_n e^{2\pi inz})(\overline{a_n} e^{-2\pi in\overline{z}}) = \int_0^1 |f(z)|^2 \, dz$$

by Parseval's identity.

AK: Is that dz? Are you integrating with respect to z?

PW: Yeah?... Oh wait, I want this to be independent of z... *I then mistakenly change dz to dy.*

AK: No, that should be dx.

PW: Oh, yeah; okay. Right. *I change the dy to dx.*

PW: So wait... Now we know that $f \in S_k$, so it has exponential decay as $y \to \infty$; thus, $\left|f(z)\, y^k\right| \leq C$. And so, $|f(z)| \ll y^{-k}$. Or, I guess we want that to be $|f(z)| \ll y^{-k/2}$ (the same argument holds).

PW: So we have that,

$$\sum_{n \leq N} |a_n|^2 e^{-4\pi n y} = \int_0^1 |f(z)|^2 \, dx \ll y^{-k}.$$

HI: *Makes some comment to another committee member about "letting me cook" or "he's cooking."*

PW: And then with the choice $y = 1/N$ we have

$$|a_N|^2 \leq \sum_{n \leq N} |a_n|^2 \leq e^{4\pi N y} \sum_{n \leq N} |a_n|^2 e^{-4\pi n y} \ll e^{4\pi N y} y^{-k} \ll N^k$$

which is exactly what we want.

AK: And what is the geometric interpretation of this integral in the Hecke bound?

PW: Of the integral?

AK: Yeah. I suppose this isn't fair because its not on your syllabus, but what is the line you're integrating over here? Write out $z = x + iy$.

PW: *I write $z = x + iy$.*

AK: Okay, and what are x and y here. Draw a graph with the fundamental domain $\mathcal{H}\backslash\Gamma$ on it and show me where this line is.

PW: *I draw a graph with the fundamental domain, and a line at $(0, 1) + i/N$.*

AK: So this is a low-lying horocycle, and you're integrating over this low-lying horocycle.

PW: Ah, okay. I see.

HI: And what is the bound that Deligne proved?

PW: It's $(k-1)/2$ instead of $k/2$.

HI: What about the constant?

PW: It's $2n^{(k-1)/2}$.

HI: For primes yes.

PW: Yeah, for primes.

HI: More generally, its a divisor function.

PW: Yes.

MY: So for Eisenstein series, why doesn't this Hecke bound work?

PW: Well, I guess one of our assumptions in this argument was that f is a cusp form, and the Eisenstein series is not since the constant term is non-zero.

MY: Sure; that means your proof doesn't work, but why can't it work?

PW: I'm not sure.

MY: Okay. What's the Fourier expansion of E_k?

PW: Okay, we have that

$$E_k(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, z)^{-k} = 1 - \frac{2k}{B_k} \sum_{1 \leq n} \sigma_{k-1}(n)\, e(nz).$$

MY: Alright, and what is $\sigma_{k-1}(n)$?

PW: *I give the definition of $\sigma_{k-1}(n)$ and add the words "I guess, right?" to the end.*

MY: And what do we know about the growth of $\sigma_{k-1}(n)$?

PW: Well I guess the $d = n$ term grows like n^{k-1} which is worse than $n^{k/2}$.

MY: Which is worse than $n^{k/2}$.

PW: Yeah that is worse. Okay.

MY: So it can't work.

(This part is a little fuzzy, but we start discussing the space of oldforms, the space of newforms, and newforms).

HI: Okay can you define a newform?

PW: Sure... I guess I need to start first by defining the space of newforms and then we can build up to that. So, we have that if $f \in S_k(\Gamma_0(M))$ then $f \in S_k(\Gamma_0(N))$ if $M \mid N$. Similarly, we if $f(z) \in S_k(\Gamma_0(M))$ then $f(rz) \in S_k(\Gamma_0(rM))$. So we define oldforms as those "coming from lower level" via one of these maps. Specifically

$$S_k^{\text{old}}(\Gamma_0(N)) = \bigcup_{\substack{M|N \\ M \neq N}} \bigcup_{r|(N/M)} \{f(rz) : f \in S_k(\Gamma_0(M))\}.$$

And then the space of newforms is the orthogonal complement of this w.r.t the Petersson inner product. Yeah. And then a newform is a normalized eigenform of all the Hecke operators coprime to the level.

HI: *Expresses some dissatisfaction with my answer (he wanted me to also state that newforms lie in the space of newforms) and my use of the term "the space of newforms."*

MY: *Expresses some agreement about how the term is a misnomer but widespread.*

AK: So why do we know that newforms exist?

PW: Because of the self-adjointness of Hecke operators coprime to the level w.r.t the Petersson inner product and the fact that Hecke operators commute.

AK: Could you elaborate on that?

PW: Sure, since Hecke operators are self adjoint w.r.t the Petersson inner product, we know that the eigenvalues have to be real. This in turn gives us that any two eigenforms with different eigenvalues are orthogonal, so you can decompose the space of newforms into a direct sum of the eigenspaces.

HI: But this doesn't give you that they're simultaneously diagonalizable.

PW: True. To get that you need the commutativity of the Hecke operators; since they commute, you can show that if you do this decomposition w.r.t one operator, then the others preserve those eigenspaces as well.

HI: Okay great.

AK: Okay great. Does any one have any more questions? I think we're ready to make a decision.

(The exam ends after ~ 58.5 minutes with a pass after the deliberation period)

HI: You're a free man now...