Oral Qualifying Exam Study Guide

Preston Tranbarger

Fall Semester 2025

Contents

1 Analytic Number Theory

1.1 Possion Summation and Mellin Transform

Definition 1.1 (Fourier Transform)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be integrable. We define the Fourier transform of f as*

$$\hat{f}(s) = \int_{\mathbb{R}} f(t)\, e^{-2\pi i s t}\, dt.$$

Lemma 1.2 (Common Fourier Transforms)**.** *We have the following common Fourier transforms*

$f(t)$	$\hat{f}(s)$
$g(n+t)$	$e^{2\pi i n s}\, \hat{g}(s)$
$g(nt)$	$\dfrac{\hat{g}(s/n)}{n}$
$e^{int}\, g(t)$	$\hat{g}\left(s - \dfrac{n}{2\pi}\right)$
$\hat{g}(t)$	$g(-s)$
$(g * h)(t)$	$\hat{g}(s)\, \hat{h}(s)$
$g(t)\, h(t)$	$(\hat{g} * \hat{h})(s)$
1	$\delta(s)$
$\delta(t)$	1

Note that here, convolution is given by

$$(g * h)(t) = \int_{-\infty}^{\infty} g(s)\, h(t-s)\, ds.$$

Remark. *In some sense, convolution of Fourier transforms is an additive rule. Whereas with Mellin transforms, it it is a multiplicative rule.*

Definition 1.3 (Schwartz Function)**.** *A function $f \in C^{\infty}(\mathbb{R})$ is Schwartz if for all m and n we have*

$$\left|\frac{d^m f}{dx^m}\right| \ll |x|^{-n} \qquad as \qquad x \to \pm\infty.$$

Theorem 1.4 (Poisson Summation)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be integrable and Schwartz. It follows that*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

Proof. Let us define

$$F(x) = \sum_{n \in \mathbb{Z}} f(x+n).$$

Note that this function is 1-periodic since it is a sum over $\mathbb{Z}$. We take its Fourier series expansion

$$F(x) = \sum_{m \in \mathbb{Z}} a_m e^{2\pi i m x} \qquad \text{with} \qquad a_m = \int_0^1 F(x)\, e^{-2\pi i m x}\, dx.$$

Now note that

$$a_m = \int_0^1 \sum_{n \in \mathbb{Z}} f(x+n)\, e^{-2\pi i m x}\, dx = \sum_{n \in \mathbb{Z}} \int_0^1 f(x+n)\, e^{-2\pi i m x}\, dx;$$

here we can interchange the integration and summation since $(0,1)$ is finite w.r.t Lebesgue measure. Note that for all $m,\, n \in \mathbb{Z}$ we have $e^{-2\pi i m n} = 1$. So it follows by translation invariance of Lesbegue measure

$$a_m = \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n)\, e^{-2\pi i m x}\, dx = \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n)\, e^{-2\pi i m(x+n)}\, d(x+n).$$

Now since f is Schwarz, we have absolute convergence; thus,

$$a_m = \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n)\, e^{-2\pi i m(x+n)}\, d(x+n) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i m x}\, dx = \hat{f}(m).$$

Thus we have

$$F(x) = \sum_{m\in\mathbb{Z}} \hat{f}(m)\, e^{2\pi i m x}$$

and specializing $x = 0$ completes the proof. □

Definition 1.5 (Mellin Transform)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be integrable. We define the Mellin transform of f as*

$$(\mathcal{M}f)(s) = \int_0^\infty t^{s-1}\, f(t)\, dt.$$

Additionally, we define the inverse Mellin transform as

$$(\mathcal{M}^{-1}f)(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s}\, f(s)\, ds$$

independent of $c \in \mathbb{R}$.

Theorem 1.6 (Mellin Inversion Theorem)**.** *If f is analytic in the strip $a < \sigma < b$ and tends to zero uniformly as $\mathfrak{Im}(s) \to \pm\infty$, then for*

$$g = (\mathcal{M}^{-1}f)(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s} f(s)\, ds$$

we have that

$$f = (\mathcal{M}g)(s) = \int_0^\infty t^{s-1}\, g(t)\, dt.$$

Lemma 1.7 (Common Mellin Transforms)**.** *We have the following common Fourier transforms*

$f(t)$	$(\mathcal{M}f)(s)$
$t^n\, g(t)$	$(\mathcal{M}g)(s+n)$
$g(1/t)$	$(\mathcal{M}g)(-s)$
$(g * h)(t)$	$(\mathcal{M}g)(s) \cdot (\mathcal{M}h)(s)$
$g(t) \cdot h(t)$	$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (\mathcal{M}g)(t) \cdot (\mathcal{M}h)(s-t)\, dt$
e^{-t}	$\Gamma(s)$
$\delta(t-n)$	n^{t-1}

Note that here, convolution is given by

$$(g * h)(t) = \int_{-\infty}^{\infty} g(s)\, h(t/s)\, ds/s.$$

1.2 Dirichlet Characters

Definition 1.8 (Dirichlet Characters)**.** *Let $\chi : \mathbb{Z} \to \mathbb{C}$ be a Dirichlet character with associated modulus q satisfying*

$$\chi(ab) = \chi(a)\,\chi(b) \qquad \text{and} \qquad \chi(a) \neq 0 \iff (a, q) = 1 \qquad \text{and} \qquad \chi(a + q) = \chi(a)$$

for all a and b.

Definition 1.9 (Primitive Dirichlet Characters)**.** *We say that χ has a quasiperiod d if $\chi(m) = \chi(n)$ for all m and n coprime to q such that $m \equiv n \mod d$. The smallest quasiperiod of χ is its conductor. If a character's conductor equals its modulus, it is primitive.*

Lemma 1.10 (Orthogonality of Dirichlet Characters I)**.** *For a character χ with modulo q, we have that*

$$\sum_{m=0}^{q-1} \chi(m) = \begin{cases} \phi(q) & \chi = \chi_0 \\ 0 & \chi \neq \chi_0. \end{cases}$$

Proof. The proof of the first case is trivial: if $\chi = \chi_0$ then there are $\phi(q)$ summands which are 1 and all other summands are 0. In the other case, there exists some $a \in (\mathbb{Z}/q\mathbb{Z})^\times$ such that $\chi(a) \neq 1$. Noting that $m \mapsto am$ is a bijective map on $\mathbb{Z}/q\mathbb{Z}$ we have that

$$\sum_{m=0}^{q-1} \chi(m) = \sum_{m=0}^{q-1} \chi(am) = \chi(a) \sum_{m=0}^{q-1} \chi(m) \implies (\chi(a) - 1) \sum_{m=0}^{q-1} \chi(m) = 0.$$

Since $\chi(a) - 1 \neq 0$, it must be that our sum over m is exactly 0. □

Lemma 1.11 (Orthogonality of Dirichlet Characters II)**.** *As a sum over characters with modulo q, we have that*

$$\sum_{\chi} \chi(a) = \begin{cases} \phi(q) & a \equiv 1 \mod q \\ 0 & a \not\equiv 1 \mod q. \end{cases}$$

Proof. The proof of the first case is trivial, since the group of characters with modulo q is isomorphic to $(\mathbb{Z}/q\mathbb{Z})^\times$, we know there are $\phi(q)$ terms of the form $\chi(1) = 1$. In the other case, there exists some χ' such that $\chi'(a) \neq 1$. Noting that $\chi \mapsto \chi\chi'$ is a bijective map on the group of characters with modulo q we have that

$$\sum_{\chi} \chi(a) = \sum_{\chi} \chi\chi'(a) = \chi'(a) \sum_{\chi} \chi(a) \implies (\chi'(a) - 1) \sum_{\chi} \chi(a) = 0.$$

Since $\chi'(a) - 1 \neq 0$, it must be that the sum over χ is exactly 0. □

Definition 1.12 (Gauss Sums)**.** *For a character χ with modulo q (usually we only care about the primitive ones) we define the Gauss sum of frequency $n \in \mathbb{Z}$, $\tau_n(\chi)$ as*

$$\tau_n(\chi) = \sum_{m=0}^{q-1} \chi(m)\, e^{2\pi i mn/q}.$$

For notation sake, we define the basic Gauss sum as $\tau(\chi) = \tau_1(\chi)$.

Lemma 1.13. *If χ is a primitive character with modulo q and $(n, q) = 1$, then we have that $\overline{\chi}(n)\,\tau(\chi) = \tau_n(\chi)$*

Proof. Note that $m \mapsto mn$ is a bijective map in $\mathbb{Z}/q\mathbb{Z}$. So we have that

$$\overline{\chi}(n)\,\tau(\chi) = \overline{\chi}(n) \sum_{m=0}^{q-1} \chi(m)\, e^{2\pi i m/q} = \overline{\chi}(n) \sum_{m=0}^{q-1} \chi(mn)\, e^{2\pi i mn/q} = \sum_{m=0}^{q-1} \chi(m)\, e^{2\pi i mn/q} = \tau_n(\chi)$$

since $\overline{\chi}(n)\,\chi(n) = 1$. □

Lemma 1.14. *If χ is a primitive character with modulo q then $\tau(\chi)\tau(\overline{\chi}) = \chi(-1)\,q$.*

Proof. Note that

$$\tau(\chi)\,\tau(\overline{\chi}) = \sum_{m=0}^{q-1} \overline{\chi}(m)\,\tau(\chi)\,e^{2\pi i m/q} = \sum_{m=0}^{q-1} \tau_m(\chi)\,e^{2\pi i m/q}$$
$$= \sum_{m=0}^{q-1}\sum_{n=0}^{q-1} \chi(n)\,e^{2\pi i m n/q}\,e^{2\pi i m/q} = \sum_{n=0}^{q-1}\chi(n)\sum_{m=0}^{q-1} e^{2\pi i m(n+1)/q}.$$

Now by orthogonality relations, we know that the inner sum vanishes if $n \not\equiv -1 \bmod q$ and equals q if $n \equiv -1 \bmod q$. Thus, the result follows as everything but the $m = q-1$ term vanishes. □

Lemma 1.15. *If χ is a primitive character with modulo q, then we have that*

$$\chi(n) = \frac{1}{q}\sum_{m=0}^{q-1} \tau_{-m}(\chi)\,e^{2\pi i m n/q}.$$

Proof. Note that

$$\sum_{m=0}^{q-1} \tau_{-m}(\chi)\,e^{2\pi i m n/q} = \sum_{m=0}^{q-1}\sum_{j=0}^{q-1} \chi(j)\,e^{-2\pi i m j/q}\,e^{2\pi i m n/q} = \sum_{j=0}^{q-1}\chi(j)\sum_{m=0}^{q-1} e^{2\pi i m(n-j)/q}.$$

Now by orthogonality relations, we know the inner sum vanishes if $j \not\equiv n \bmod q$ and equals q if $j \equiv n \bmod q$. Thus, everything but the $j = n$ term vanishes

$$\sum_{m=0}^{q-1} \tau_{-m}(\chi)\,e^{2\pi i m n/q} = \sum_{j=0}^{q-1}\chi(j)\sum_{m=0}^{q-1} e^{2\pi i m(n-j)/q} = \chi(n)\,q.$$

Dividing through by q completes the proof. □

Definition 1.16 (Root Number of a Character)**.** *Given a primitive character χ, we define the root number of a character $W(\chi)$ as*

$$W(\chi) = \frac{\tau(\chi)}{i^{\delta}\sqrt{q}} \qquad \text{where} \qquad \delta = \begin{cases} 0 & \chi(-1) = 1 \\ 1 & \chi(-1) = -1. \end{cases}$$

Lemma 1.17. *We have that $W(\chi)\,W(\overline{\chi}) = 1$.*

Proof. We have that

$$W(\chi)\,W(\overline{\chi}) = \frac{\tau(\chi)\,\tau(\overline{\chi})}{(-1)^{\delta} q} = \frac{\chi(-1)\,q}{(-1)^{\delta} q}.$$

Since $\chi(-1) = (-1)^{\delta}$, everything cancels. □

1.3 Riemann ζ Function and Dirichlet L-Function

Definition 1.18 (Riemann ζ Function)**.** *We define $\zeta(s)$ for $\mathfrak{Re}(s) > 1$ as the sum*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Definition 1.19 (Dirichlet L-Function)**.** *For χ a Dirichlet character, we define $L(\chi, s)$ for $\mathfrak{Re}(s) > 1$ as the sum*

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Lemma 1.20 (Euler Product of ζ)**.** *For $\mathfrak{Re}(s) > 1$ we have that*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - 1/p^s}.$$

Lemma 1.21 (Euler Product of L-functions)**.** *For $\mathfrak{Re}(s) > 1$ we have that*

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \chi(p)/p^s}.$$

Lemma 1.22 (Equivalent Form of ζ)**.** *Let*

$$\zeta_0(s) = \frac{s}{s-1} - s \int_1^{\infty} \{x\}\, x^{-s} \, \frac{dx}{x}.$$

We have that $\zeta(s) = \zeta_0(s)$ for $\sigma > 1$.

Proof. Note that for $\sigma > 1$ we have

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{n}{n^s} - \sum_{n=1}^{\infty} \frac{n-1}{n^s} = \sum_{n=1}^{\infty} \frac{n}{n^s} - \sum_{n=0}^{\infty} \frac{n}{(n+1)^s} = \sum_{n=1}^{\infty} \frac{n}{n^s} - \sum_{n=1}^{\infty} \frac{n}{(n+1)^s}.$$

Thus

$$\zeta(s) = \sum_{n=1}^{\infty} n\,(n^{-s} - (n+1)^{-s}).$$

Now we note that

$$s \int_n^{n+1} x^{-s} \, \frac{dx}{x} = s \left(-\frac{1}{s}\, x^{-s} \right)_n^{n+1} = n^{-s} - (n+1)^{-s}.$$

So, substituting this we have

$$\zeta(s) = \sum_{n=1}^{\infty} n\,(n^{-s} - (n+1)^{-s}) = s \sum_{n=1}^{\infty} n \int_n^{n+1} x^{-s} \, \frac{dx}{x} = s \int_1^{\infty} \lfloor x \rfloor \, x^{-s} \, \frac{dx}{x}.$$

But noting that $\lfloor x \rfloor = x - \{x\}$ we have that

$$\zeta(s) = s \int_1^{\infty} \lfloor x \rfloor \, x^{-s} \, \frac{dx}{x} = s \int_1^{\infty} x^{-s} \, dx - s \int_1^{\infty} \{x\}\, x^{-s} \, \frac{dx}{x}.$$

Evaluating the first integral completes the result. □

Corollary 1.23 (Meromorphic Continuation of ζ to $\sigma > 0$)**.** *We have that $\zeta_0(s)$ has a simple pole at $s = 1$ and is meromorphic on the half plane $\sigma > 0$.*

Proof. Note that for $\sigma > 0$ we have

$$\left| s \int_1^{\infty} \{x\}\, x^{-s} \, \frac{dx}{x} \right| \le |s| \int_1^{\infty} \left| \{x\}\, x^{-s-1} \right| dx \le |s| \int_1^{\infty} x^{-\sigma-1} \, dx = \frac{|s|}{\sigma}.$$

So this integral converges and we have that $\zeta_0(s)$ is meromorphic on the half plane $\sigma > 0$.

Additionally, it follows that $\zeta_0(s)$ has a simple pole at $s = 1$. □

Theorem 1.24 (Analytic Continuation of ζ)**.** *We can analytically continue ζ to the entire complex plane.*

Proof. Recall that for $\sigma > 0$, $\Gamma(s)$ is defined as

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt.$$

Now if we do the substitution $t = nu$, then after some manipulation we have that

$$n^{-s}\,\Gamma(s) = \int_0^\infty e^{-nt} t^{s-1}\, dt.$$

Now we take the sum over all $n \geq 1$.

$$\zeta(s)\,\Gamma(s) = \sum_{1\leq n} n^{-s}\,\Gamma(s) = \sum_{1\leq n} \int_0^\infty e^{-nt} t^{s-1}\, dt.$$

Now note that, using the substitution $u = nt$ we have

$$\sum_{1\leq n} \int_0^\infty \left|e^{-nt} t^{s-1}\right| dt = \sum_{1\leq n} \int_0^\infty e^{-nt} t^{\sigma-1}\, dt = \sum_{1\leq n} \frac{1}{n^\sigma} \int_0^\infty e^{-u} u^{\sigma-1}\, du = \Gamma(\sigma)\,\zeta(\sigma).$$

For $\sigma > 1$ we know this converges. So by Fubini-Tonelli, when $\sigma > 1$, we can interchange the sum and integral as needed.

$$\zeta(s)\,\Gamma(s) = \sum_{1\leq n} \int_0^\infty e^{-nt} t^{s-1}\, dt = \int_0^\infty \sum_{1\leq n} e^{-nt} t^{s-1}\, dt = \int_0^\infty \frac{t^{s-1}e^{-t}}{1-e^{-t}}\, dt.$$

Now using the substitution $t = 2u$ we have

$$\zeta(s)\,\Gamma(s) = \int_0^\infty \frac{t^{s-1}e^{-t}}{1-e^{-t}}\, dt = 2\int_0^\infty \frac{(2u)^{s-1}e^{-2u}}{1-e^{-2u}}\, du = 2^s \int_0^\infty \frac{u^{s-1}e^{-2u}}{1-e^{-2u}}\, du.$$

Thus,

$$(1-2^{1-s})\,\zeta(s)\,\Gamma(s) = \int_0^\infty t^{s-1}\left(\frac{e^{-t}}{1-e^{-t}} - \frac{2e^{-2t}}{1-e^{-2t}}\right) dt = \int_0^\infty \frac{t^{s-1}}{1+e^t}\, dt.$$

Using integration by parts with $u = 1/(1+e^t)$ and $dv = t^{s-1}$, we have that

$$(1-2^{1-s})\,\zeta(s)\,\Gamma(s) = \int_0^\infty \frac{t^{s-1}}{1+e^t}\, dt = \left(\frac{t^s}{s\,(1+e^t)}\right)\Bigg|_0^\infty + \frac{1}{s}\int_0^\infty \frac{t^s e^t}{(1+e^t)^2}\, dt.$$

Re-arranging we have

$$(1-2^{1-s})\,\zeta(s)\,\Gamma(s+1) = \int_0^\infty \frac{t^s e^t}{(1+e^t)^2}\, dt \implies \zeta(s) = \frac{1}{(1-2^{1-s})\,\Gamma(s+1)} \int_0^\infty \frac{t^s e^t}{(1+e^t)^2}\, dt.$$

Using integration by parts with $u = e^t/(1+e^t)^2$ and $dv = t^s$, we have that

$$(1-2^{1-s})\,\zeta(s)\,\Gamma(s+1) = \int_0^\infty \frac{t^s e^t}{(1+e^t)^2}\, dt = \left(\frac{t^{s+1}e^t}{(s+1)(1+e^t)^2}\right)\Bigg|_0^\infty + \frac{1}{s+1}\int_0^\infty \frac{t^{s+1}(e^{2t}-e^t)}{(1+e^t)^3}\, dt.$$

Re-arranging we have

$$(1-2^{1-s})\,\zeta(s)\,\Gamma(s+2) = \int_0^\infty \frac{t^{s+1}(e^{2t}-e^t)}{(1+e^t)^3}\, dt \implies \zeta(s) = \frac{1}{(1-2^{1-s})\,\Gamma(s+2)} \int_0^\infty \frac{t^{s+1}(e^{2t}-e^t)}{(1+e^t)^3}\, dt.$$

Repeating this procedure k-times yeilds an expression of the form

$$\zeta(s) = \frac{(-1)^k}{(1-2^{1-s})\,\Gamma(s+k)} \int_0^\infty t^{s+k-1}\left(\frac{d^k}{dt^k}\,\frac{1}{1+e^t}\right) dt$$

Note that this expression is analytic for $\sigma > -k$ for all k (except for the pole at $s = 1$); thus, we can extend $\zeta(s)$ to the entire plane analytically (except for the pole at $s = 1$). □

Lemma 1.25 ($\zeta(s)$ Zeros I)**.** *$\zeta(s)$ has trivial zeros at the negative even integers.*

Proof. Note that from the previous proof, by the fundamental theorem of calculus, we have that

$$\zeta(1-k) = \frac{(-1)^k}{(1-2^k)} \int_0^\infty \left(\frac{d^k}{dt^k} \frac{1}{1+e^t} \right) dt = \frac{(-1)^k}{(1-2^k)} \left(\frac{d^{k-1}}{dt^{k-1}} \frac{1}{1+e^t} \right) \Bigg|_0^\infty = -\frac{(-1)^k}{(1-2^k)} \left(\frac{d^{k-1}}{dt^{k-1}} \frac{1}{1+e^t} \right) \Bigg|_{t=0}.$$

One finds that

$$\frac{1}{1+e^t} = \frac{1}{2} + \sum_{1 \le k} \frac{(1-2^{k+1})B_{k+1}x^k}{(k+1)!}.$$

Thus for $1 \le k$ we have

$$\zeta(1-k) = -\frac{(-1)^k (k-1)!}{(1-2^k)} \left(\frac{(1-2^k)B_k}{k!} \right) = -\frac{(-1)^k B_k}{k}.$$

So, if $1-k$ is even then k is odd and $B_k = 0$; thus, $\zeta(1-k) = 0$ also. ◻

Lemma 1.26 ($\zeta(s)$ Poles)**.** *$\zeta(s)$ has a pole at $s = 1$ with residue 1.*

Proof. Note that for fixed $s \in \mathbb{R}$, t^{-s} is a monotone decreasing function. Thus we have that

$$(n+1)^{-s} < \int_n^{n+1} t^{-s}\, dt < n^{-s}.$$

Summing over all $n \ge 1$ we have

$$\zeta(s) - 1 < \int_1^\infty t^{-s}\, dt < \zeta(s).$$

Noting that the integral evaluates as $(s-1)^{-1}$, rearranging gives us that $1 < (s-1)\,\zeta(s) < s$. Letting $s \to 1$ from above gives the desired result. ◻

Definition 1.27 (Jacobi θ function)**.** *We define the Jacobi θ function as follows. Let*

$$\theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = 1 + 2 \sum_{1 \le n} e^{-\pi n^2 t}.$$

Lemma 1.28 (Functional Equation for θ)**.** *We have that*

$$\theta(1/t) = \sqrt{t}\, \theta(t).$$

Proof. Note that via Poisson summation we have

$$\theta(1/t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2/t} = \sum_{m \in \mathbb{Z}} \int_{\mathbb{R}} e^{-\pi n^2/t} e^{-2\pi i m n}\, dn.$$

By using the substitution $n = u\sqrt{t}$ we have that

$$\begin{aligned} \int_{\mathbb{R}} e^{-\pi n^2/t} e^{-2\pi i m n}\, dn &= \sqrt{t} \int_{\mathbb{R}} \exp\Big(-\pi u^2 - 2\pi i m u \sqrt{t} \Big)\, du \\ &= \sqrt{t} \int_{\mathbb{R}} \exp\Big(-\pi (u^2 + 2 i m u \sqrt{t} - m^2 t + m^2 t) \Big)\, du \\ &= e^{-\pi m^2 t} \sqrt{t} \int_{\mathbb{R}} \exp\Big(-\pi (u + i m \sqrt{t})^2 \Big)\, du = e^{-\pi m^2 t} \sqrt{t}. \end{aligned}$$

Thus we have that

$$\theta(1/t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2/t} = \sum_{m \in \mathbb{Z}} \int_{\mathbb{R}} e^{-\pi n^2/t} e^{-2\pi i m n}\, dn = \sqrt{t} \sum_{m \in \mathbb{Z}} e^{-\pi m^2 t} = \sqrt{t}\, \theta(t).$$

◻

Theorem 1.29 (Functional Equation for ζ)**.** *We have that*

$$\frac{\Gamma(s/2)\,\zeta(s)}{\pi^{s/2}} = \frac{\Gamma((1-s)/2)\,\zeta(1-s)}{\pi^{(1-s)/2}}.$$

Proof. Recall the definition of the Γ function and substitute $x = \pi n^2 t$:

$$\Gamma(s/2) = \int_0^\infty e^{-x} x^{s/2-1}\,dx = \pi^{s/2} n^s \int_0^\infty e^{-\pi n^2 t} t^{s/2-1}\,dt.$$

Thus

$$\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) = \sum_{1\le n} n^{-s}\pi^{-s/2}\,\Gamma(s/2) = \sum_{1\le n}\int_0^\infty e^{-\pi n^2 t} t^{s/2-1}\,dt.$$

Now note that, using the substitution $u = \pi n^2 t$ we have

$$\sum_{1\le n}\int_0^\infty \left| e^{-\pi n^2 t} t^{s/2-1}\right| dt = \sum_{1\le n}\int_0^\infty e^{-\pi n^2 t} t^{\sigma/2-1}\,dt = \pi^{-\sigma/2}\sum_{1\le n}\frac{1}{n^\sigma}\int_0^\infty e^{-u}\,u^{\sigma/2-1}\,du = \frac{\zeta(\sigma)\,\Gamma(\sigma/2)}{\pi^{\sigma/2}}.$$

For $\sigma > 1$ we know this converges. So by Fubini-Tonelli, when $\sigma > 1$, we can interchange the sum and integral as needed.

$$\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) = \sum_{1\le n}\int_0^\infty e^{-\pi n^2 t} t^{s/2-1}\,dt = \int_0^\infty \sum_{1\le n} e^{-\pi n^2 t} t^{s/2-1}\,dt = \frac{1}{2}\int_0^\infty (\theta(t)-1)\,t^{s/2-1}\,dt.$$

Applying linearity, we note that

$$\int_0^\infty (\theta(t)-1)\,t^{s/2-1}\,dt = -\frac{2}{s} + \int_0^1 \theta(t)\,t^{s/2-1}\,dt + \int_1^\infty (\theta(t)-1)\,t^{s/2-1}\,dt.$$

Focusing on the first integral and using the substitution $t = 1/u$, we have

$$\int_0^1 \theta(t)\,t^{s/2-1}\,dt = -\int_\infty^1 \theta(1/u)\,u^{-s/2-1}\,du = \int_1^\infty \theta(u)\,u^{(1-s)/2-1}\,du.$$

via the functional equation for θ. Thus,

$$\int_0^1 \theta(t)\,t^{s/2-1}\,dt = \int_1^\infty \theta(t)\,t^{(1-s)/2-1}\,dt = \int_1^\infty (\theta(t)-1)\,t^{(1-s)/2-1}\,dt + \frac{2}{s-1}.$$

So we have that

$$\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) = -\frac{1}{s\,(1-s)} + \int_1^\infty (\theta(t)-1)\left(t^{s/2-1} + t^{(1-s)/2-1}\right)\,dt.$$

Noting that the RHS is equivalent under the substitution $s \leftrightarrow 1-s$ completes the proof. □

Lemma 1.30 ($\zeta(s)$ Zeros II)**.** *Every non-trivial zeta zero lies in the strip* $0 < \sigma < 1$.

Proof. Let $\sigma < 0$ such that s is not a trivial zeta zero and $\zeta(s) = 0$. Then $1-\sigma > 1$, and from the Euler product expansion of ζ we know that $\zeta(1-s) \neq 0$. So, from the functional equation, we have that

$$0 = \zeta(s) = \pi^{s-1/2}\left(\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\right)\zeta(1-s).$$

Now recall Euler's reflection formula and Legendre's duplication formula which state

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin\pi z} \qquad \text{and} \qquad \Gamma(z)\,\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\;\Gamma(2z)$$

respectively. Substituting $z = s/2$ into the first, and $z = (1-s)/2$ into the second, we have

$$\Gamma(s/2)\,\Gamma(1-s/2) = \frac{\pi}{\sin(\pi s/2)} \qquad \text{and} \qquad \Gamma((1-s)/2)\,\Gamma(1-s/2) = 2^s\sqrt{\pi}\,\Gamma(1-s).$$

Thus,

$$\frac{\Gamma((1-s)/2)}{\Gamma(s/2)} = \frac{2^s\,\Gamma(1-s)\sin(\pi s/2)}{\sqrt{\pi}}.$$

So we have that,

$$0 = \zeta(s) = \left(\frac{(2\pi)^s\ \Gamma(1-s)\sin(\pi s/2)}{\pi}\right)\zeta(1-s).$$

Since $\zeta(1-s) \neq 0$, and noting that $\Gamma(1-s) \neq 0$ also; it must be the case that $\sin(\pi s/2) = 0$. Thus, $s = -2, -4, -6, \ldots$.

Therefore, any other zeros must lie in the strip $0 < \sigma < 1$. □

Theorem 1.31 (Analytic Continuation of L-functions)**.** *We can analytically continue $L(\chi, s)$ to the entire complex plane.*

Proof. Recall that for all $\sigma > 0$, $\Gamma(s)$ is defined as

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\,dt.$$

Now if we do the substitution $t = nu$, then after some manipulation we have that

$$n^{-s}\,\Gamma(s) = \int_0^\infty e^{-nt} t^{s-1}\,dt.$$

Now if we take the sum over all $n \geq 1$ with character χ.

$$L(\chi, s)\,\Gamma(s) = \sum_{1\leq n}\chi(n)\,n^{-s}\,\Gamma(s) = \sum_{1\leq n}\chi(n)\int_0^\infty e^{-nt}\,t^{s-1}\,dt.$$

Now note that, using the substitution $u = nt$ we have

$$\sum_{1\leq n}\int_0^\infty \left|\chi(n)\,e^{-nt}\,t^{s-1}\right| dt \leq \sum_{1\leq n}\int_0^\infty e^{-nt}\,t^{\sigma-1}\,dt = \sum_{1\leq n}\frac{1}{n^\sigma}\int_0^\infty e^{-u}\,u^{\sigma-1}\,du = \Gamma(\sigma)\,\zeta(\sigma).$$

For $\sigma > 1$ we know this converges. So by Fubini-Tonelli, when $\sigma > 1$, we can interchange the sum and integral as needed.

$$L(\chi, s)\,\Gamma(s) = \sum_{1\leq n}\chi(n)\int_0^\infty e^{-nt}t^{s-1}\,dt = \int_0^\infty \sum_{1\leq n}\chi(n)\,e^{-nt}t^{s-1}\,dt.$$

We now break up our sum modulo q where $n = j + mq$,

$$L(\chi, s)\,\Gamma(s) = \int_0^\infty \sum_{1\leq n}\chi(n)\,e^{-nt}t^{s-1}\,dt = \sum_{1\leq j\leq q}\chi(j)\int_0^\infty \sum_{0\leq m} e^{-(j+mq)\,t}t^{s-1}\,dt = \sum_{1\leq j\leq q}\chi(j)\int_0^\infty \frac{t^{s-1}\,e^{-jt}}{1-e^{-qt}}\,dt.$$

Now using the substitution $t = 2u$ we have

$$L(\chi, s)\,\Gamma(s) = \sum_{1\leq j\leq q}\chi(j)\int_0^\infty \frac{t^{s-1}e^{-jt}}{1-e^{-qt}}\,dt = 2\sum_{1\leq j\leq q}\chi(j)\int_0^\infty \frac{(2u)^{s-1}e^{-2ju}}{1-e^{-2qu}}\,du = 2^s\sum_{1\leq j\leq q}\chi(j)\int_0^\infty \frac{u^{s-1}\,e^{-2ju}}{1-e^{-2qu}}\,du.$$

Thus,

$$\begin{aligned}(1-2^{1-s})\,L(\chi, s)\,\Gamma(s) &= \sum_{1\leq j\leq q}\chi(j)\int_0^\infty t^{s-1}\left(\frac{e^{-jt}}{1-e^{-qt}} - \frac{2e^{-2jt}}{1-e^{-2qt}}\right)dt\\ &= \sum_{1\leq j\leq q}\chi(j)\int_0^\infty t^{s-1}\left(\frac{e^{-jt}(1-2e^{-jt}+e^{-qt})}{1-e^{-2qt}}\right)dt.\end{aligned}$$

Now note that if $t = 0$ then $e^{-t} = 1$. So, $1 - 2e^{-jt} + e^{-qt} = 0$ and $1 - e^{-2qt} = 0$. Thus we can factor a $1 - e^{-t}$ term out of the numerator and denominator to get

$$(1-2^{1-s})\,L(\chi, s)\,\Gamma(s) = \sum_{1\leq j\leq q}\chi(j)\int_0^\infty t^{s-1}\left(\frac{P_j(e^{-t})}{1+e^{-t}+\ldots+e^{-(2q-1)t}}\right)dt$$

where P_j is some degree $j+q-1$ polynomial. Using repeated integration by parts and rearranging gives us an expression of the form

$$L(\chi,s)=\frac{(-1)^k}{(1-2^{1-s})\,\Gamma(s+k)}\sum_{1\le j\le q}\chi(j)\int_0^\infty t^{s+k-1}\,Q_{j,k}(e^{-t})\,dt$$

where $Q_{j,k}$ is some quotient of polynomials with $Q_{j,k}(e^{-t})$ having exponential decay at ∞.

Note that this expression is analytic for $\sigma > -k$ for all k; thus, we can extend $L(\chi,s)$ to the entire plane analytically. □

Lemma 1.32 ($L(\chi,s)$ Negative Values)**.** *For $k\ge 1$ we have that*

$$L(\chi,1-k)=-\frac{(-1)^k B_{k,\chi}}{k}\qquad \textit{where}\qquad \sum_{1\le j\le q}\chi(j)\frac{te^{jt}}{e^{qt}-1}=\sum_{0\le n}\frac{B_{n,\chi}}{n!}\,t^n.$$

Proof. (Sketch) Note that by the fundamental theorem of calculus and the previous equation we have that

$$L(\chi,1-k)=\frac{(-1)^k}{(1-2^k)}\sum_{1\le j\le q}\chi(j)\int_0^\infty Q_{j,k}(e^{-t})\,dt=-\frac{(-1)^k}{(1-2^k)}\sum_{1\le j\le q}\chi(j)\left(\frac{d^{k-1}}{d^{k-1}}\frac{P_j(e^{-t})}{1+e^{-t}+\ldots+e^{-(2q-1)t}}\right)\Bigg|_{t=0}.$$

By evaluating the power series expansion of

$$\frac{P_j(e^{-t})}{1+e^{-t}+\ldots+e^{-(2q-1)t}}=\sum_{1\le k}c_{j,k}x^k$$

then for $1\le k$ we have

$$L(\chi,1-k)=-\frac{(-1)^k}{(1-2^k)}\sum_{1\le j\le q}\chi(j)\left((k-1)!\,c_{j,k-1}\right).$$

Actually evaluating $c_{j,k-1}$ and substituting yields the desired result. □

Definition 1.33 (Twisted θ Function)**.** *Given a primitive character χ with modulo q, we define $\delta=0$ if $\chi(-1)=1$ and $\delta=1$ if $\chi(-1)=-1$. Now we define the θ function twisted by the character χ as*

$$\theta_\chi(t)=\sum_{n\in\mathbb{Z}}n^\delta\chi(n)\,e^{-\pi n^2t/q}=2\sum_{1\le n}n^\delta\chi(n)\,e^{-\pi n^2t/q}.$$

Lemma 1.34 (Functional Equation for Twisted θ Functions)**.** *We have that*

$$\theta_\chi(1/t)=W(\chi)\,t^{1/2+\delta}\,\theta_{\overline{\chi}}(t).$$

Proof. Recall that we can express $\chi(n)$ as a linear combination of $\tau_{-m}(\chi)$ with m ranging over $\mathbb{Z}/q\mathbb{Z}$. Thus we write

$$\theta_\chi(1/t)=\sum_{m=0}^{q-1}\frac{\tau_{-m}(\chi)}{q}\sum_{n\in\mathbb{Z}}n^\delta\,e^{2\pi imn/q-\pi n^2/(qt)}. \tag{1}$$

We will revisit this equation later. By Poisson summation, and using the substitution $n=u\sqrt{r}$ we have that

$$\sum_{n\in\mathbb{Z}}e^{-\pi\,(x+n)^2/r}=\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}e^{-\pi(x+n)^2/r}e^{-2\pi imn}\,dn=\sqrt{r}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}e^{-\pi(u+x/\sqrt{r})^2}e^{-2\pi imu\sqrt{r}}\,dr.$$

Now note that

$$-\pi\left(u+\frac{x}{\sqrt{r}}\right)^2-2\pi imu\sqrt{r}=-\pi\left(u+\frac{x}{\sqrt{r}}+im\sqrt{r}\right)^2+2\pi ixm-\pi m^2r,$$

so we have that

$$\sum_{n\in\mathbb{Z}}e^{-\pi(x+n)^2/r}=\sqrt{r}\sum_{m\in\mathbb{Z}}\exp\bigl(2\pi ixm-\pi m^2r\bigr)\int_{\mathbb{R}}\exp\left(-\pi\left(u+\frac{x}{\sqrt{r}}+im\sqrt{r}\right)^2\right)\,dr.$$

The inner integral is trivially 1. So after some manipulation we have

$$\sum_{n\in\mathbb{Z}} e^{2\pi ixn-\pi n^2 r} = r^{-1/2}\sum_{n\in\mathbb{Z}} e^{-\pi(x+n)^2/r}.$$

Differentiating term by term w.r.t the variable x (we can do this because of the exponential decay in the terms as $n \to \pm\infty$) and rearranging slightly we have

$$\sum_{n\in\mathbb{Z}} ne^{2\pi ixn-\pi n^2 r} = ir^{-3/2}\sum_{n\in\mathbb{Z}}(x+n)\, e^{-\pi(x+n)^2/r}.$$

So, if we let $x = m/q$ and $r = 1/(qt)$ then by algebraic manipulation we have

$$\sum_{n\in\mathbb{Z}} n^\delta e^{2\pi imn/q-\pi n^2/(qt)} = i^\delta (qt)^{1/2+\delta}\sum_{n\in\mathbb{Z}}(m/q+n)^\delta e^{-\pi(m/q+n)^2 qt}.$$

So returning to 1 we have that

$$\begin{aligned}\theta_\chi(1/t) &= \frac{i^\delta\,(qt)^{1/2+\delta}}{q}\sum_{m=0}^{q-1}\tau_{-m}(\chi)\sum_{n\in\mathbb{Z}}(m/q+n)^\delta e^{-\pi(m/q+n)^2 qt}\\ &= (it)^\delta\sqrt{\frac{t}{q}}\sum_{m=0}^{q-1}\tau_{-m}(\chi)\sum_{n\in\mathbb{Z}}(m+nq)^\delta e^{-\pi(m+nq)^2 t/q} = (it)^\delta\sqrt{\frac{t}{q}}\sum_{n\in\mathbb{Z}} n^\delta\tau_{-n}(\chi)\, e^{-\pi n^2 t/q}.\end{aligned}$$

Recalling that $\overline{\chi}(-n)\,\tau(\chi) = \tau_{-n}(\chi)$, noting that $\overline{\chi}(-n) = (-1)^\delta\overline{\chi}(n)$, and $-i = 1/i$; we now have that

$$\theta_\chi(1/t) = \tau(\chi)\left(\frac{t}{i}\right)^\delta\sqrt{\frac{t}{q}}\sum_{n\in\mathbb{Z}} n^\delta\,\overline{\chi}(n)\, e^{-\pi n^2 t/q} = W(\chi)\, t^{1/2+\delta}\,\theta_{\overline{\chi}}(t)$$

exactly as desired. □

Theorem 1.35 (Functional Equation for L-functions)**.** *For a primitive character χ with modulo q and $\chi(-1) = (-1)^\delta$, we have that*

$$\left(\frac{q}{\pi}\right)^{s/2}\Gamma((s+\delta)/2)\, L(\chi,s) = W(\chi)\,\left(\frac{q}{\pi}\right)^{(1-s)/2}\Gamma((1-s+\delta)/2)\, L(\overline{\chi},1-s)$$

Proof. Recall the definition of the Γ function and substitute $x = \pi n^2 t/q$:

$$\Gamma((s+\delta)/2) = \int_0^\infty e^{-x}x^{(s+\delta)/2-1}\,dx = n^s\left(\frac{\pi}{q}\right)^{(s+\delta)/2}\int_0^\infty n^\delta e^{-\pi n^2 t/q}t^{(s+\delta)/2-1}\,dt.$$

Thus

$$\left(\frac{q}{\pi}\right)^{(s+\delta)/2}\Gamma((s+\delta)/2)\, L(\chi,s) = \sum_{1\le n}\frac{\chi(n)}{n^s}\left(\frac{q}{\pi}\right)^{(s+\delta)/2}\Gamma((s+\delta)/2) = \sum_{1\le n}\int_0^\infty \chi(n)\, n^\delta e^{-\pi n^2 t/q}t^{(s+\delta)/2-1}\,dt.$$

Now note that, using the substitution $u = \pi n^2 t/q$ we have

$$\begin{aligned}\sum_{1\le n}\int_0^\infty\left|\chi(n)\, n^\delta\, e^{-\pi n^2 t/q}t^{(s+\delta)/2-1}\right|dt &\le \sum_{1\le n}\int_0^\infty n^\delta\, e^{-\pi n^2 t/q}t^{(\sigma+\delta)/2-1}\,dt\\ &= \left(\frac{q}{\pi}\right)^{(\sigma+\delta)/2}\sum_{1\le n}\frac{1}{n^\sigma}\int_0^\infty e^{-u}u^{(\sigma+\delta)/2-1}\,du = \left(\frac{q}{\pi}\right)^{(\sigma+\delta)/2}\Gamma((\sigma+\delta)/2)\,\zeta(\sigma).\end{aligned}$$

For $\sigma > 1$ we know this converges. So by Fubini-Tonelli, when $\sigma > 1$, we can interchange the sum and integral as needed.

$$\begin{aligned}\left(\frac{q}{\pi}\right)^{(s+\delta)/2}\Gamma((s+\delta)/2)\, L(\chi,s) &= \sum_{1\le n}\int_0^\infty\chi(n)\, n^\delta\, e^{-\pi n^2 t/q}t^{(s+\delta)/2-1}\,dt\\ &= \int_0^\infty\sum_{1\le n}\chi(n)\, n^\delta\, e^{-\pi n^2 t/q}t^{(s+\delta)/2-1}\,dt = \frac{1}{2}\int_0^\infty\theta_\chi(t)\, t^{(s+\delta)/2-1}\,dt.\end{aligned}$$

Applying linearity, we note that

$$\int_0^\infty \theta_\chi(t)\, t^{(s+\delta)/2-1}\, dt = \int_0^1 \theta_\chi(t)\, t^{(s+\delta)/2-1}\, dt + \int_1^\infty \theta_\chi(t)\, t^{(s+\delta)/2-1}\, dt.$$

Focusing on the first integral and using the substitution $t = 1/u$, we have

$$\int_0^1 \theta_\chi(t)\, t^{(s+\delta)/2-1}\, dt = -\int_\infty^1 \theta_\chi(1/u)\, u^{-(s+\delta)/2-1}\, du = W(\chi) \int_1^\infty \theta_{\overline{\chi}}(u)\, u^{(1-s+\delta)/2-1}\, du$$

via the functional equation for θ_χ. So we have that

$$2\left(\frac{q}{\pi}\right)^{(s+\delta)/2} \Gamma((s+\delta)/2)\, L(\chi, s) = W(\chi) \int_1^\infty \theta_{\overline{\chi}}(t)\, t^{(1-s+\delta)/2-1}\, dt + \int_1^\infty \theta_\chi(t)\, t^{(s+\delta)/2-1}\, dt.$$

By replacing χ with $\overline{\chi}$ and s with $1-s$ we have

$$2\left(\frac{q}{\pi}\right)^{(1-s+\delta)/2} \Gamma((1-s+\delta)/2)\, L(\overline{\chi}, 1-s) = W(\overline{\chi}) \int_1^\infty \theta_{\overline{\chi}}(t)\, t^{(s+\delta)/2-1}\, dt + \int_1^\infty \theta_\chi(t)\, t^{(1-s+\delta)/2-1}\, dt.$$

Multiplying through by $W(\chi)$ and recalling that $W(\chi)\, W(\overline{\chi}) = 1$, we have

$$2\, W(\chi)\left(\frac{q}{\pi}\right)^{(1-s+\delta)/2} \Gamma((1-s+\delta)/2)\, L(\overline{\chi}, 1-s) = \int_1^\infty \theta_\chi(t)\, t^{(s+\delta)/2-1}\, dt + W(\chi) \int_1^\infty \theta_{\overline{\chi}}(t)\, t^{(1-s+\delta)/2-1}\, dt.$$

Thus,

$$2\left(\frac{q}{\pi}\right)^{(s+\delta)/2} \Gamma((s+\delta)/2)\, L(\chi, s) = 2\, W(\chi)\left(\frac{q}{\pi}\right)^{(1-s+\delta)/2} \Gamma((1-s+\delta)/2)\, L(\overline{\chi}, 1-s).$$

Removing common factors on both sides proves the desired result. □

Lemma 1.36 ($L(\chi, s)$ Zeros)**.** *Every non-trivial zero of $L(\chi, s)$ lies in the strip $0 < \sigma < 1$.*

Proof. Let $\sigma < 0$ such that $L(\chi, s) = 0$. Then $1 - \sigma > 1$, and from the Euler product expansion of L-functions we know that $L(\overline{\chi}, 1-s) \neq 0$. So, from the functional equation, we have that

$$0 = L(\chi, s) = W(\chi)\left(\frac{\pi^{s-1/2}\, \Gamma((1-s+\delta)/2)}{q^{s-1/2}\, \Gamma((s+\delta)/2)}\right) L(\overline{\chi}, 1-s).$$

Now recall Euler's reflection formula and Legendre's duplication formula which state

$$\Gamma(z)\, \Gamma(1-z) = \frac{\pi}{\sin \pi z} \qquad \text{and} \qquad \Gamma(z)\, \Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\; \Gamma(2z)$$

respectively. If $\delta = 0$ then substituting $z = s/2$ into the first, and $z = (1-s)/2$ into the second we have

$$\Gamma(s/2)\, \Gamma(1-s/2) = \frac{\pi}{\sin(\pi s/2)} \qquad \text{and} \qquad \Gamma((1-s)/2)\, \Gamma(1-s/2) = 2^s\sqrt{\pi}\; \Gamma(1-s).$$

Thus,

$$\frac{\Gamma((1-s+\delta)/2)}{\Gamma((s+\delta)/2)} = \frac{\Gamma((1-s)/2)\, \Gamma(1-s/2)}{\Gamma(s/2)\, \Gamma(1-s/2)} = \frac{2^s\, \Gamma(1-s)\sin(\pi s/2)}{\sqrt{\pi}}.$$

So we have that,

$$0 = L(\chi, s) = W(\chi)\left(\frac{(2\pi)^s\, \Gamma(1-s)\sin(\pi s/2)}{\pi q^{s-1/2}}\right) L(\overline{\chi}, 1-s).$$

Since $L(\overline{\chi}, 1-s) \neq 0$, and noting that $\Gamma(1-s) \neq 0$ also; it must be the case that $\sin(\pi s/2) = 0$. Thus $s = -2, -4, -6, \ldots$.

Therefore, any other zeros must lie in the strip $0 < \sigma < 1$.

If $\delta = 1$ then substituting $z = (1-s)/2$ into the first, and $z = (1-s)/2$ into the second we have

$$\Gamma((1-s)/2)\, \Gamma((1+s)/2) = \frac{\pi}{\sin(\pi(1-s)/2)} \qquad \text{and} \qquad \Gamma((1-s)/2)\, \Gamma((2-s)/2) = 2^s\sqrt{\pi}\, \Gamma(1-s).$$

Thus,

$$\frac{\Gamma((1-s+\delta)/2)}{\Gamma((s+\delta)/2)} = \frac{\Gamma((1-s)/2)\,\Gamma((2-s)/2)}{\Gamma((1-s)/2)\,\Gamma((1+s)/2)} = \frac{2^s\,\Gamma(1-s)\sin(\pi(1-s)/2)}{\sqrt{\pi}} = \frac{2^s\,\Gamma(1-s)\cos(\pi s/2)}{\sqrt{\pi}}.$$

So we have that,

$$0 = L(\chi, s) = W(\chi)\left(\frac{(2\pi)^s\,\Gamma(1-s)\cos(\pi s/2)}{\pi q^{s-1/2}}\right) L(\overline{\chi}, 1-s).$$

Since $L(\overline{\chi}, 1-s) \neq 0$, and noting that $\Gamma(1-s) \neq 0$ also; it must be the case that $\cos(\pi s/2) = 0$. Thus $s = -1, -3, -5, \ldots$.

Therefore, any other zeros must lie in the strip $0 < \sigma < 1$. $\square$

1.4 Dirichlet's Theorem on Primes in Arithmetic Progressions & PNT

Theorem 1.37 (Infinitely Many Primes from $\zeta(s)$)**.** *There exist infinitely many primes.*

Proof. Let $s > 1$ be real. For all questions in this proof regarding convergence, it suffices to check that $|1/p^s| < 1$ for all primes p. From Lemma 1.20 we know that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - 1/p^s}.$$

So by taking the logarithm of both sides, we know that

$$\log \zeta(s) = -\sum_p \log(1 - 1/p^s).$$

Now recall that for $|z| < 1$ we have that $-\log(1-z) = z + z^2/2 + z^3/3 + \ldots$. Now we have that

$$\log \zeta(s) = \sum_p \sum_{1 \le n} \frac{1}{np^{ns}} = \sum_p \frac{1}{p^s} + \sum_p \sum_{2 \le n} \frac{1}{np^{ns}}. \tag{2}$$

But now note that

$$\sum_p \sum_{2 \le n} \frac{1}{np^{ns}} \le \sum_p \sum_{2 \le n} \frac{1}{p^{ns}} = \sum_p \frac{p^{-2s}}{1 - p^{-s}} = \sum_p \frac{1}{p^s\left(p^s - 1\right)} = \sum_p \left(\frac{1}{p^s - 1} - \frac{1}{p^s}\right) \le \sum_p \left(\frac{1}{p-1} - \frac{1}{p}\right).$$

Continuing we have that

$$\sum_p \sum_{2 \le n} \frac{1}{np^{ns}} \le \sum_p \left(\frac{1}{p-1} - \frac{1}{p}\right) \le \sum_{2 \le n} \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1.$$

Now returning to (2) we have that

$$\log \zeta(s) \le \sum_p \frac{1}{p^s} + 1.$$

If we suppose that there are finitely many primes then we have that $\lim_{s \to 1^+} \zeta(s) < \infty$; however, this directly contradicts Lemma 1.26. So, there must be infinitely many primes. $\square$

Theorem 1.38 (Non-Vanishing of $L(\chi, 1)$ implies Dirichlet's Theorem)**.** *If $L(\chi, 1) \neq 0$ for all Dirichlet characters $\chi \neq \chi_0$, then there exist infinitely many primes $p \equiv a \bmod q$ when $(a, q) = 1$.*

Proof. Let $s > 1$ be real. For all questions in this proof regarding convergence, it suffices to check that $|1/p^s| < 1$ for all primes p. From Lemma 1.21 we know that

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \chi(p)/p^s}.$$

So by taking the logarithm of both sides, we know that

$$\log L(\chi, s) = -\sum_p \log(1 - \chi(p)/p^s).$$

Now recall that for $|z| < 1$ we have that $-\log(1-z) = z + z^2/2 + z^3/3 + \ldots$. Now we have that

$$\log L(\chi, s) = \sum_p \sum_{1 \le n} \frac{\chi(p^n)}{np^{ns}}.$$

By the above we have that

$$\frac{1}{\varphi(q)} \sum_{\chi \in X_q} \overline{\chi}(a) \log L(\chi, s) = \frac{1}{\varphi(q)} \sum_{\chi \in X_q} \overline{\chi}(a) \sum_p \sum_{1 \le n} \frac{\chi(p^n)}{np^{ns}} = \sum_p \sum_{1 \le n} \frac{1}{np^{ns}} \left(\frac{1}{\varphi(q)} \sum_{\chi \in X_q} \overline{\chi}(a)\chi(p^n)\right).$$

Now applying Lemma 1.11 we have that

$$\frac{1}{\varphi(q)} \sum_{\chi \in X_q} \overline{\chi}(a) \log L(\chi, s) = \sum_{1 \leq n} \left(\sum_{\substack{p \\ p \equiv a \bmod q}} \frac{1}{np^{ns}} \right) = \sum_{\substack{p \\ p \equiv a \bmod q}} \frac{1}{p^s} + \sum_{2 \leq n} \left(\sum_{\substack{p \\ p^n \equiv a \bmod q}} \frac{1}{np^{ns}} \right). \tag{3}$$

But now using a similar argument as in Theorem 1.37 we know that

$$\sum_{2 \leq n} \left(\sum_{\substack{p \\ p^n \equiv a \bmod q}} \frac{1}{np^{ns}} \right) \leq 1.$$

Thus, we have that

$$\lim_{s \to 1^+} \frac{1}{\varphi(q)} \sum_{\chi \in X_q} \overline{\chi}(a) \log L(\chi, s) = \infty \Longleftrightarrow \sum_{\substack{p \\ p \equiv a \bmod q}} \frac{1}{p^s} = \infty \Longleftrightarrow \text{Dirichlet's theorem.} \tag{4}$$

Now note that

$$\lim_{s \to 1^+} \frac{1}{\varphi(q)} \sum_{\chi \in X_q} \overline{\chi}(a) \log L(\chi, s) = \lim_{s \to 1^+} \frac{\overline{\chi_0}(a)}{\varphi(q)} \log \left(\zeta(s) \prod_{p | q} (1 - 1/p^s) \right) + \lim_{s \to 1^+} \frac{1}{\varphi(q)} \sum_{\substack{\chi \in X_q \\ \chi \neq \chi_0}} \overline{\chi}(a) \log L(\chi, s).$$

Now note that $\lim_{s \to 1^+}(1 - 1/p^s) = 1 - 1/p$ and $\lim_{s \to 1^+} \zeta(s) = \infty$ by Lemma 1.26. Thus by (4) we have

$$\left| \lim_{s \to 1^+} \frac{1}{\varphi(q)} \sum_{\substack{\chi \in X_q \\ \chi \neq \chi_0}} \overline{\chi}(a) \log L(\chi, s) \right| < \infty \implies \text{Dirichlet's Theorem.}$$

But now note that if $L(\chi, 1) \neq 0$ for all $\chi \neq \chi_0$ then the left hand side of the above must be true. So we have that

$$L(\chi, 1) \neq 0 \text{ for all } \chi \neq \chi_0 \implies \text{Dirichlet's theorem.}$$

□

Lemma 1.39 (Proof that $L(\chi, 1) \neq 0$ for χ Complex)**.** *If χ is complex then $L(\chi, 1) \neq 0$.*

Proof. Now recalling equation (3) and substituting $a = 1$ we have

$$\lim_{s \to 1^+} \frac{1}{\varphi(q)} \sum_{\chi \in X_q} \log L(\chi, s) = \lim_{s \to 1^+} \sum_{1 \leq n} \left(\sum_{\substack{p \\ p^n \equiv 1 \bmod q}} \frac{1}{np^{ns}} \right)$$
$$\implies \lim_{s \to 1^+} \sum_{\chi \in X_q} \log L(\chi, s) \geq 0 \implies \lim_{s \to 1^+} \prod_{\chi \in X_q} L(\chi, s) \geq 1.$$

Now suppose that χ is complex and $L(\chi, 1) = 0$. Then we have that

$$\lim_{s \to 1^+} \prod_{\chi \in X_q} L(\chi, s) = \lim_{s \to 1^+} L(\chi_0, s) \prod_{\substack{\chi \in X_q \\ \chi \neq \chi_0}} L(\chi, s) = \lim_{s \to 1^+} \zeta(s) \left(\prod_{p | q} (1 - 1/p^s) \right) \prod_{\substack{\chi \in X_q \\ \chi \neq \chi_0}} L(\chi, s) = 0.$$

This is because $L(\chi, 1) = L(\overline{\chi}, 1) = 0$, $\zeta(s)$ has a simple pole at $s = 1$ by Lemma 1.26, and every other term is analytic at $s = 1$. This of course is a contradiction. □

Lemma 1.40 (Elementary Proof that $L(\chi,1)\neq 0$ for χ Real)**.** *If χ is real then $L(\chi,1)\neq 0$.*

Proof. (See Davenport Chapter 4 pp 33-34.)

Now suppose that χ is real and $L(\chi,1)=0$. Let us define

$$\psi(s)=\frac{L(\chi,s)\,L(\chi_0,s)}{L(\chi_0,2s)}.$$

Note that the numerator is analytic on the region $\mathfrak{Re}(s)>0$ since by assumption $L(\chi,1)=0$ cancels with the simple pole of $L(\chi_0,s)$ at $s=1$. The denominator is non-zero and analytic on the region $\mathfrak{Re}(s)>1/2$. Thus, ψ is analytic on the region $\mathfrak{Re}(s)>1/2$. Additionally, since $L(\chi_0,2s)\to\infty$ as $s\to 1/2$ we note that $\psi(s)\to 0$ as $s\to 1/2$.

Now note that we have the Euler product expansion for $\psi(s)$ as

$$\psi(s)=\prod_p\frac{1-\chi_0(p)\,p^{-2s}}{(1-\chi(p)\,p^{-s})(1-\chi_0(p)\,p^{-s})}.$$

Now if $p\mid q$, then $\chi_0(p)=\chi(p)=0$. So,

$$\frac{1-\chi_0(p)\,p^{-2s}}{(1-\chi(p)\,p^{-s})(1-\chi_0(p)\,p^{-s})}=1.$$

If $p\nmid q$, then $\chi_0(p)=1$ and $\chi(p)=\pm1$ since χ is real. If $\chi(p)=-1$, then

$$\frac{1-\chi_0(p)\,p^{-2s}}{(1-\chi(p)\,p^{-s})(1-\chi_0(p)\,p^{-s})}=\frac{1-p^{-2s}}{(1+p^{-s})(1-p^{-s})}=1.$$

Alternatively; if $\chi(p)=1$, then

$$\frac{1-\chi_0(p)\,p^{-2s}}{(1-\chi(p)\,p^{-s})(1-\chi_0(p)\,p^{-s})}=\frac{1-p^{-2s}}{(1-p^{-s})(1-p^{-s})}=\frac{1+p^{-s}}{1-p^{-s}}.$$

So we can write the Euler product expansion for $\psi(s)$ as

$$\psi(s)=\prod_p\frac{1-\chi_0(p)\,p^{-2s}}{(1-\chi(p)\,p^{-s})(1-\chi_0(p)\,p^{-s})}=\prod_{\substack{p\nmid q\\ \chi(p)=1}}\frac{1+p^{-s}}{1-p^{-s}}.$$

And now on the region $\mathfrak{Re}(s)>1$ we have

$$\psi(s)=\prod_{\substack{p\nmid q\\ \chi(p)=1}}\frac{1+p^{-s}}{1-p^{-s}}=\prod_{\substack{p\nmid q\\ \chi(p)=1}}(1+p^{-s})(1+p^{-s}+p^{-2s}+\ldots)=\sum_{1\le n}a_n n^{-s}$$

where $a_1=1$ and $a_n\ge 0$ for all $n\ge 1$. Thus $\psi(s)\ge 1$ for $s>1$.

Alternatively, since ψ is analytic on $\mathfrak{Re}(s)>1/2$ there exists a power series expansion about $s=2$ with radius of convergence at least $3/2$.

$$\psi(s)=\sum_{0\le m}\frac{\psi^{(m)}(2)}{m!}\,(s-2)^m.$$

However, recalling our Dirichlet series we have that

$$\psi^{(m)}(2)=(-1)^m\sum_{1\le n}\frac{a_n(\log n)^m}{n^2}=(-1)^m b_m$$

where $b_m\ge 0$. Thus,

$$\psi(s)=\sum_{0\le m}\frac{b_m}{m!}\,(2-s)^m$$

for $|2-s|<3/2$. If $1/2<s<2$ then note that we have

$$\psi(s)\ge\psi(2)\ge 1.$$

But this contradicts the earlier fact that $\psi(s)\to 0$ as $s\to 1/2$. So $L(\chi,1)\neq 0$ as desired. $\square$

Lemma 1.41 (Algebraic Proof that $L(\chi, 1) \neq 0$)**.** *For all non-trivial Dirichlet characters we have that $L(\chi, 1) \neq 0$.*

Proof. (See Iwaniec-Kowalski Chapter 2 pp 38.) Sketch:
The product

$$\prod_{\chi \in X_q} L(\chi, s) = \zeta_K(s)$$

is the Dedekind zeta function of $K =$ the cyclotomic integers at $\varphi(q)$-th roots of unity. By the class number formula we know $\lim_{s \to 1^+} (s-1)\, \zeta_K(s) \neq 0$ (note here the pole at $s = 1$ comes from $L(\chi_0, s)$); so no $L(\chi, 1)$ can vanish for non-trivial $\chi \neq \chi_0$ because then this limit would be 0. ☐

Definition 1.42 (Von Mangoldt Function)**.** *We define the Von Mangoldt function, denoted Λ, as*

$$\Lambda(n) = \begin{cases} \log p & n = p^k \text{ with } p \text{ prime} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1.43. *We have that*

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{1 \leq n} \frac{\Lambda(n)}{n^s}.$$

Proof. Using Lemma 1.20 we have that

$$\log \zeta(s) = -\sum_p \log(1 - 1/p^s).$$

By taking the derivative of both sides we have

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} = \frac{d}{ds} \log \zeta(s) &= -\sum_p \frac{\log(p)}{1 - p^s} \\ &= -\sum_p \log(p) \left(\frac{1}{1 - 1/p^s} - 1 \right) = -\sum_p \sum_{1 \leq n} \frac{\log(p)}{p^{ns}} = -\sum_{1 \leq n} \frac{\Lambda(n)}{n^s} \end{aligned}$$

☐

Lemma 1.44 (Non-Vanishing of $\zeta(1 + it)$)**.** *We have that*

$$\zeta(1 + it) \neq 0 \text{ for all } t \in \mathbb{R}$$

Proof. Let

$$F_t(\sigma) = \zeta(\sigma)^3\, \zeta(\sigma + it)^2\, \zeta(\sigma - it)^2\, \zeta(\sigma + 2it)\, \zeta(\sigma - 2it).$$

Note that since $\zeta(\overline{s}) = \overline{\zeta(s)}$ we have that $F_t(\sigma)$ is real valued. Using the Euler product expansion for $\sigma > 1$, we now examine the series expansion of $\log F_t(\sigma)$.

$$\log F_t(\sigma) = \sum_p \sum_{1 \leq n} \frac{3 + 2p^{-it} + 2p^{it} + p^{-2it} + p^{2it}}{np^{ns}}.$$

But now note that $3 + 2p^{-it} + 2p^{it} + p^{-2it} + p^{2it} = (1 + p^{it} + p^{-it})^2$. Since p^{it} and p^{-it} are conjugate, we know that $1 + p^{it} + p^{-it}$ is real. Thus $3 + 2p^{-it} + 2p^{it} + p^{-2it} + p^{2it}$ is real and non-negative. By extension

$$\log F_t(\sigma) = \sum_p \sum_{1 \leq n} \frac{3 + 2p^{-it} + 2p^{it} + p^{-2it} + p^{2it}}{np^{ns}} \geq 0 \implies F_t(\sigma) \geq 1. \tag{5}$$

Now suppose that there exists real $t \neq 0$ such that $\zeta(1 + it) = 0$. Then we have that $\zeta(1 - it) = 0$ also. Now, since ζ is analytic at $1 + it$ and $1 - it$ we have that $\zeta(\sigma + it) = O(\sigma - 1)$ and $\zeta(\sigma - it) = O(\sigma - 1)$ as $\sigma \to 1^+$. Additionally, since ζ has an order 1 pole at $s = 1$, we have that $\zeta(\sigma) = O((\sigma - 1)^{-1})$ as $\sigma \to 1^+$. Noting that there can not be a pole at $\zeta(\sigma \pm 2it)$, we immediately have that $\lim_{\sigma \to 1^+} F_t(\sigma) = 0$. This is a contradiction. Thus, $\zeta(1 + it) \neq 0$. ☐

Lemma 1.45 (Bound on $(\zeta'/\zeta)(s)$)**.** *For $|t| > 3$ there exists a constant $C \in (0, 1/2)$ such that*

$$\left| \frac{\zeta'}{\zeta} \right| \ll \log^2 |t| \qquad \text{where} \qquad \sigma > 1 - C/\log |t|.$$

Theorem 1.46 (Chebyshev Formulation of PNT)**.** *We have that*

$$\psi(x) = \sum_{n \le x} \Lambda(n) = x + O\left(x \exp\left(-c\sqrt{\log x}\right)\right)$$

where c is some constant. Here ψ is usually called the Chebyshev function.

Proof. Fix some smooth non-negative χ with compact support in $[1/2,\, 2]$ such that

$$\widetilde{\chi}(0) = \int_0^\infty \chi(y)\, \frac{dy}{y} = 1.$$

Additionally, note that by IBP we have that

$$\widetilde{\chi}(s) = \int_0^\infty \chi(y)\, y^s\, \frac{dy}{y} = \left(\frac{\chi(y)\, y^s}{s}\right)_0^\infty + \frac{1}{s}\int_0^\infty \chi'(y)\, y^s\, dy = \frac{1}{s}\int_{1/2}^2 \chi'(y)\, y^s\, dy \ll \frac{1}{s}.$$

Additionally,

$$\begin{aligned}\widetilde{\chi}(s) &= \int_0^\infty \chi(y) \exp(s \log y)\, \frac{dy}{y} \\ &= \int_0^\infty \chi(y)(1 + s\log y + O(s^2))\, \frac{dy}{y} \\ &= \int_0^\infty \chi(y)\, \frac{dy}{y} + s\int_0^\infty \chi(y)\log(y)\, \frac{dy}{y} + O(s^2) = \widetilde{\chi}(0) + s\int_{1/2}^2 \chi(y)\log(y)\, \frac{dy}{y} + O(s^2) = 1 + O(s).\end{aligned}$$

Now fix $\varepsilon > 0$ and let $\chi_\varepsilon(y) = \chi(y^{1/\varepsilon})/\varepsilon$. Then note that χ_ε is non-negative with compact support in $[2^{-\varepsilon}, 2^{\varepsilon}]$ and

$$\int_0^\infty \chi_\varepsilon(y)\, \frac{dy}{y} = \frac{1}{\varepsilon}\int_0^\infty \chi(y^{1/\varepsilon})\, \frac{dy}{y} = \int_0^\infty \chi(u)\, \frac{du}{u} = 1;$$

this uses the substitution $u = y^{1/\varepsilon}$ which gives $\varepsilon\, du/u = dy/y$. Now let $\varphi = \mathbf{1}_{x<1}$ and let $\varphi_\varepsilon = \varphi * \chi_\varepsilon$ by Mellin convolution. We then have that $\varphi_\varepsilon(x) = 1$ for $x < 2^{-\varepsilon}$ and $\varphi_\varepsilon(x) = 0$ for $x > 2^{\varepsilon}$ with some smooth behaviour on the interval $[2^{-\varepsilon}, 2^{\varepsilon}]$.

Now we switch focus. By the inverse Mellin transform we have that

$$\frac{1}{2\pi i}\int_{(2)} \left(-\frac{\zeta'}{\zeta}(s)\right) \widetilde{\varphi_\varepsilon}(s)\, x^s\, ds = \sum_n \Lambda(n)\, \varphi_\varepsilon(n/x) = \psi(x) + O(\varepsilon x \log x) \tag{6}$$

since $\Lambda(n) < \log x$ for $n < x$ and $\varphi_\varepsilon(n/x)$ differs from $\varphi(n/x)$ by less than 1 on an interval of length εx up to constant. This is because $2^{\varepsilon} - 2^{-\varepsilon}$ behaves like $2\varepsilon \log 2$ around $\varepsilon = 0$.

Now we pull contours. This gives us a new contour of integration, call this γ, which is such that we have the nice bound on ζ'/ζ along γ in accordance with Lemma 1.45. Pay attention to the fact that we must now include the residue of the pole at $s = 1$. Thus,

$$\frac{1}{2\pi i}\int_{(2)} \left(-\frac{\zeta'}{\zeta}(s)\right) \widetilde{\varphi_\varepsilon}(s)\, x^s\, ds = \widetilde{\varphi_\varepsilon}(1)\, x^1 + \frac{1}{2\pi i}\int_\gamma \left(-\frac{\zeta'}{\zeta}(s)\right) \widetilde{\varphi_\varepsilon}(s)\, x^s\, ds.$$

But now note that $\widetilde{\varphi_\varepsilon}(1) = \widetilde{\varphi}(1) \cdot \widetilde{\chi}(\varepsilon) = 1 \cdot (1 + O(\varepsilon)) = 1 + O(\varepsilon)$ since $\widetilde{\varphi}(s) = 1/s$ and $\widetilde{\chi}(0) = 1$. Thus,

$$\frac{1}{2\pi i}\int_{(2)} \left(-\frac{\zeta'}{\zeta}(s)\right) \widetilde{\varphi_\varepsilon}(s)\, x^s\, ds = x + \frac{1}{2\pi i}\int_\gamma \left(-\frac{\zeta'}{\zeta}(s)\right) \widetilde{\varphi_\varepsilon}(s)\, x^s\, ds + O(\varepsilon x). \tag{7}$$

Now we focus on our integral over γ. Let $T > 1$, we break it up into three separate integrals

$$\int_\gamma = \int_{-\infty}^{-T} + \int_{-T}^{T} + \int_T^\infty.$$

We focus on the third integral first. Note that

$$\left|\int_T^\infty \left(-\frac{\zeta'}{\zeta}(s)\right) \widetilde{\varphi_\varepsilon}(s)\, x^s\, ds\right| \le \int_T^\infty \left|-\frac{\zeta'}{\zeta}(s)\right| \widetilde{\varphi_\varepsilon}(s)\, x^\sigma\, ds \ll \int_T^\infty \log^2 t \left(\frac{1}{\varepsilon t^2}\right) x\, dt \ll \frac{x}{\varepsilon}\int_T^\infty \frac{\sqrt{t}\, dt}{t^2} \ll \frac{x}{\varepsilon\sqrt{T}}. \tag{8}$$

since $|-\zeta/\zeta| \ll \log^2|t|$ by Lemma 1.45, and $\widetilde{\varphi_\varepsilon}(s) = \widetilde{\varphi}(s)\cdot\widetilde{\chi_\varepsilon}(s) = \widetilde{\chi}(\varepsilon s)/s \ll 1/(\varepsilon s^2)$, and $x^\sigma < x$ for $\sigma < 1$. Following an identical procedure yields the same bound on the first integral. Now for the second integral, note that

$$\left|\int_{-T}^{T}\left(-\frac{\zeta'}{\zeta}(s)\right)\widetilde{\varphi_\varepsilon}(s)\,x^s\,ds\right| \le \int_{-T}^{T}\left|-\frac{\zeta'}{\zeta}(s)\right|\widetilde{\varphi_\varepsilon}(s)\,x^\sigma\,ds \le D\int_{-T}^{T}\left(\frac{1}{\varepsilon\,(1+t^2)}\right)x^{1-C/\log T}\,dt \ll \frac{x^{1-C/\log T}}{\varepsilon}. \quad (9)$$

where D is taken to be the supremum of $|\zeta'/\zeta|$ over the curve (which is compact, so D exists and is finite). Now putting together equations (6), (7), (8), and (9) we have that

$$\psi(x) + O(\varepsilon x\log x) = x + O(\varepsilon x) + O\left(\frac{x}{\varepsilon\sqrt{T}}\right) + O\left(\frac{x^{1-C/\log T}}{\varepsilon}\right).$$

Now we optimize in T by letting $\sqrt{T} = x^{C/\log T}$. Thus, $\log T = \sqrt{2C\log x}$. Now note that

$$x^{-C/\log T} = \exp\left(-\frac{C}{\log T}\log x\right) = \exp\left(-\frac{C}{\sqrt{2C\log x}}\log x\right) = \exp\left(-\sqrt{\frac{C\log x}{2}}\right).$$

So, by absorbing $O(\varepsilon x)$ into $O(\varepsilon x\log x)$ and substituting the above, we have

$$\psi(x) = x + O(\varepsilon x\log x) + O\left(\frac{x}{\varepsilon}\exp\left(-\sqrt{\frac{C\log x}{2}}\right)\right).$$

Now we optimize in ε by letting $\varepsilon\log x = \exp\Big(-\sqrt{(C\log x)/2}\Big)/\varepsilon$. Thus,

$$\varepsilon = \exp\left(-\sqrt{\frac{C\log x}{8}} - \frac{1}{2}\log\log x\right).$$

Now note that

$$\varepsilon x\log x = x\exp\left(-\sqrt{\frac{C\log x}{8}} - \frac{1}{2}\log\log x\right)\exp(\log\log x) = x\exp\left(-\sqrt{\frac{C\log x}{8}} + \frac{1}{2}\log\log x\right).$$

So by substituting the above, we have

$$\psi(x) = x + O\left(x\exp\Big(-c\sqrt{\log x}\Big)\sqrt{\log x}\right)$$

where $c = \sqrt{C/8}$. We can absorb the $\sqrt{\log x}$ into the exponential to get the desired result. □

Corollary 1.47 (Traditional PNT). *We have that*

$$\pi(x) \sim \frac{x}{\log x}$$

as $x \to \infty$.

Proof. Note that

$$\Pi(x) = \sum_{n\le x}\frac{\Lambda(n)}{\log n} \qquad \text{and} \qquad \frac{1}{\log n} = \frac{1}{\log x} + \left(-\frac{1}{\log t}\right)\Bigg|_n^x = \frac{1}{\log x} + \int_n^x \frac{dt}{t\log^2 t}.$$

Thus,

$$\Pi(x) = \frac{\psi(x)}{\log x} + \sum_{n\le x}\Lambda(n)\int_n^x\frac{dt}{t\log^2 t} = \frac{\psi(x)}{\log x} + \int_2^x\frac{\psi(t)\,dt}{t\log^2 t}.$$

Now by the Chebyshev formulation of the PNT, we know that for all $\epsilon > 0$ there exists x_e such that for all $x > x_\epsilon$ we have that $\psi(x) = x + O(\epsilon x)$. Thus,

$$\Pi(x) = \frac{x + O(\epsilon x)}{\log x} + \int_2^x\frac{t + O(\epsilon t)}{t\log^2 t}\,dt = (1 + O(\epsilon))\left(\frac{x}{\log x} + \int_2^x\frac{dt}{\log^2 t}\right).$$

Now note that

$$\int_2^x \frac{dt}{\log^2 t} = \int_2^{\sqrt{x}} \frac{dt}{\log^2 t} + \int_{\sqrt{x}}^x \frac{dt}{\log^2 t}$$
$$\leq \frac{(\sqrt{x}-2)}{\log^2 2} + \frac{(x-\sqrt{x})}{\log^2 \sqrt{x}} \leq \frac{\sqrt{x}}{\log^2 2} + \frac{4x}{\log^2 x} = O\left(\frac{x}{\log^2 x}\right).$$

Putting these together and letting $\epsilon \to 0$ we know that

$$\Pi(x) \sim \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

Now note that

$$\Pi(x) = \pi(x) + \frac{\pi(x^{1/2})}{2} + \frac{\pi(x^{1/3})}{3} + \ldots \leq \pi(x) + \frac{x^{1/2}}{2} + \frac{x^{1/3}}{3} + \ldots = \pi(x) + O(\sqrt{x}).$$

So the desired result follows immediately since $\sqrt{x}$ is $O(x/\log^2 x)$. ☐

Corollary 1.48 (n-th Prime Growth)**.** *We have that* $p_n \sim n \log n$.

Proof. FAKE NEWS: From the traditional PNT we know that

$$n = \pi(p_n) \sim \frac{p_n}{\log p_n}.$$

Thus,

$$\begin{aligned} p_n &\sim n \log p_n \\ &\sim n \log(n \log p_n) = n \log n + n \log \log p_n \\ &\sim n \log n + n \log \log(n \log p_n) = n \log n + n \log \log n + n \log \log \log p_n \\ &\sim \quad \vdots \\ &\sim n \log n + n \log \log n + n \log \log \log n + \ldots \end{aligned}$$

Ignoring the higher order terms, the desired result follows. ☐

Proof. LEGIT: From the traditional PNT we know that

$$1 = \lim_{n\to\infty} \frac{\pi(n)}{n \log n} = \lim_{n\to\infty} \frac{\pi(p_n)}{p_n/\log p_n} = \lim_{n\to\infty} \frac{n}{p_n/\log p_n}.$$

Thus, taking the reciprocal of the last equality and simplifying we have

$$1 = \lim_{n\to\infty} \frac{p_n}{n \log p_n} \tag{10}$$

Taking the logarithm of both sides we have

$$0 = \lim_{n\to\infty} \log p_n - \log n - \log\log p_n = \lim_{n\to\infty} \log p_n \left(1 - \frac{\log n}{\log p_n} - \frac{\log\log p_n}{\log p_n}\right) = \lim_{n\to\infty} \log p_n \left(1 - \frac{\log n}{\log p_n}\right),$$

since $\lim_{n\to\infty} \log\log p_n/\log p_n = 0$ trivially. But noting that $\lim_{n\to\infty} \log p_n \neq 0$ we must have that

$$\lim_{n\to\infty} \frac{\log n}{\log p_n} = 1.$$

And now by the above and equation (10) note that

$$\lim_{n\to\infty} \frac{p_n}{n \log n} = \lim_{n\to\infty} \frac{p_n}{n \log p_n} \cdot \frac{\log p_n}{\log n} = 1 \cdot 1 = 1.$$

☐

Corollary 1.49 (Primorial Growth). *Let P_n be the n-th primorial, the product of the first n primes:*

$$P_n = \prod_{k \le n} p_n.$$

We have that $P_n \sim \exp((1 + o(1))\, n \log n)$.

Proof. Note that

$$\sum_{p \le x} \log p = \pi(x) \log x - \sum_{p \le x} (\log x - \log p) = \pi(x) \log x - \sum_{p \le n} \int_p^x \frac{dt}{t} = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t}\, dt.$$

From the traditional PNT we know that $\pi(x) \sim x/\log x + o(x/\log x)$. Thus

$$\pi(x) \log x \sim x + o(x) \qquad \text{and} \qquad \int_2^x \frac{\pi(t)}{t}\, dt = \int_2^x \frac{1}{\log t} + o\left(\int_2^x \frac{dt}{\log t} \right) = o(x).$$

Thus $\sum_{p \le x} \log p = x + o(x) - o(x) = x + o(x)$. Now note that

$$P_n = \prod_{k \le n} p_n = \exp\left(\sum_{p \le p_n} \log p \right) = \exp(p_n + o(p_n)).$$

From the above, we know that $p_n \sim n \log n$; thus,

$$P_n \sim \exp((1 + o(1))\, n \log n)$$

as desired. □

Theorem 1.50 (Chebyshev Formulation of PNT for Arithmetic Progressions). *We have that*

$$\psi_{a,q}(x) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a) \sum_{n \le x} \chi(n)\, \Lambda(n) = \frac{x}{\varphi(q)} + O\left(x \exp\left(-c\sqrt{\log x} \right) \right)$$

where c is some constant.

Proof. Fix some smooth non-negative χ with compact support in $[1/2, 2]$ such that

$$\widetilde{\chi}(0) = \int_0^\infty \chi(y) \frac{dy}{y} = 1.$$

Additionally, note that by IBP we have that

$$\widetilde{\chi}(s) = \int_0^\infty \chi(y)\, y^s \frac{dy}{y} = \left(\frac{\chi(y)\, y^s}{s} \right)_0^\infty + \frac{1}{s} \int_0^\infty \chi'(y)\, y^s\, dy = \frac{1}{s} \int_{1/2}^2 \chi'(y)\, y^s\, dy \ll \frac{1}{s}.$$

Additionally,

$$\begin{aligned} \widetilde{\chi}(s) &= \int_0^\infty \chi(y) \exp(s \log y) \frac{dy}{y} \\ &= \int_0^\infty \chi(y)(1 + s \log y + O(s^2))\, \frac{dy}{y} \\ &= \int_0^\infty \chi(y)\, \frac{dy}{y} + s \int_0^\infty \chi(y)\, \log(y)\, \frac{dy}{y} + O(s^2) = \widetilde{\chi}(0) + s \int_{1/2}^2 \chi(y)\, \log(y)\, \frac{dy}{y} + O(s^2) = 1 + O(s). \end{aligned}$$

Now fix $\varepsilon > 0$ and let $\chi_\varepsilon(y) = \chi(y^{1/\varepsilon})/\varepsilon$. Then note that χ_ε is non-negative with compact support in $[2^{-\varepsilon}, 2^\varepsilon]$ and

$$\int_0^\infty \chi_\varepsilon(y)\, \frac{dy}{y} = \frac{1}{\varepsilon} \int_0^\infty \chi(y^{1/\varepsilon})\, \frac{dy}{y} = \int_0^\infty \chi(u) \frac{du}{u} = 1;$$

this uses the substitution $u = y^{1/\varepsilon}$ which gives $\varepsilon\, du/u = dy/y$. Now let $\varphi = \mathbf{1}_{x<1}$ and let $\varphi_\varepsilon = \varphi * \chi_\varepsilon$ by Mellin convolution. We then have that $\varphi_\varepsilon(x) = 1$ for $x < 2^{-\varepsilon}$ and $\varphi_\varepsilon(x) = 0$ for $x > 2^{\varepsilon}$ with some smooth behavior on the interval $[2^{-\varepsilon}, \varepsilon]$.

Now we switch focus. By the inverse Mellin transform we have that

$$\frac{1}{\varphi(q)} \sum_{\chi \bmod q} \frac{\overline{\chi}(a)}{2\pi i} \int_{(2)} \left(-\frac{L_\chi'}{L_\chi}(s)\right) \widetilde{\varphi_\varepsilon}(s)\, x^s ds = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a) \sum_n \chi(n)\, \Lambda(n)\, \varphi_\varepsilon(n/x) = \psi_{a,q}(x) + O(\varepsilon x \log x) \quad (11)$$

since $\Lambda(n) < \log x$ for $n < x$ and $\varphi_\varepsilon(n/x)$ differs from $\varphi(n/x)$ by less than 1 on an interval of length εx up to constant. This is because $2^\varepsilon - 2^{-\varepsilon}$ behaves like $2\varepsilon \log 2$ around $\varepsilon = 0$.

Now we pull contours. This gives us a new contour of integration, call this γ which is such that we have the nice bound on L_χ'/L_χ along γ in accordance with Lemma 1.45. Pay attention to the fact that we must now include the residue of the pole at $s = 1$. Thus

$$\frac{1}{2\pi i} \int_{(2)} \left(-\frac{L_\chi'}{L_\chi}(s)\right) \widetilde{\varphi_\varepsilon}(s)\, x^s\, ds = \widetilde{\varphi}(1)\, x^1\, \delta_{\chi,\chi_0} + \frac{1}{2\pi i} \int_\gamma \left(-\frac{L_\chi'}{L_\chi}(s)\right) \widetilde{\varphi_\varepsilon}(s)\, x^s\, ds.$$

But now note that $\widetilde{\varphi_\varepsilon}(1) = \widetilde{\varphi}(1) \cdot \widetilde{\chi}(\varepsilon) = 1 \cdot (1 + O(\varepsilon))$ since $\widetilde{\varphi}(s) = 1/s$ and $\widetilde{\chi}(0) = 1$. Thus,

$$\frac{1}{\varphi(q)} \sum_{\chi \bmod q} \frac{\overline{\chi}(a)}{2\pi i} \int_{(2)} \left(-\frac{L_\chi'}{L_\chi}(s)\right) \widetilde{\varphi_\varepsilon}(s)\, x^s\, ds = \frac{x}{\varphi(q)} + \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \frac{\overline{\chi}(a)}{2\pi i} \int_\gamma \left(-\frac{L_\chi'}{L_\chi}(s)\right) \widetilde{\varphi_\varepsilon}(s)\, x^s\, ds + O(\varepsilon x). \quad (12)$$

Now we focus on our integral over γ. Let $T > 1$, we break it up into three seperate integrals

$$\int_\gamma = \int_{-\infty}^{T} + \int_{-T}^{T} + \int_{T}^{\infty}.$$

We focus on the third integral first. Note that

$$\left|\int_T^\infty \left(-\frac{L_\chi'}{L_\chi}(s)\right) \widetilde{\varphi_\varepsilon}(s)\, x^s\, ds\right| \le \int_T^\infty \left|-\frac{L_\chi'}{L_\chi}(s)\right| \widetilde{\varphi_\varepsilon}(s)\, x^\sigma\, ds \ll \int_T^\infty \log^2 t \left(\frac{1}{\varepsilon t^2}\right) x\, dt \ll \frac{x}{\varepsilon} \int_T^\infty \frac{\sqrt{t}\, dt}{t^2} \ll \frac{x}{\varepsilon\sqrt{T}}. \quad (13)$$

since $\left|-L_\chi'/L_\chi\right| \ll \log^2|t|$ by Lemma 1.45, and $\widetilde{\varphi_\varepsilon}(s) = \widetilde{\varphi}(s) \cdot \widetilde{\chi_\varepsilon}(s) = \widetilde{\chi}(\varepsilon s)/s \ll 1/(\varepsilon s^2)$, and $x^\sigma < x$ for $\sigma < 1$. Following an identical procedure yields the same bound on the first integral. Now for the second integral, note that

$$\begin{aligned} \left|\int_{-T}^T \left(-\frac{L_\chi'}{L_\chi}(s)\right) \widetilde{\varphi_\varepsilon}(s)\, x^s\, ds\right| &\le \int_{-T}^T \left|-\frac{L_\chi'}{L_\chi}(s)\right| \widetilde{\varphi_\varepsilon}(s)\, x^\sigma\, ds \\ &\ll \int_{-T}^T \log^2(1+|t|) \left(\frac{1}{\varepsilon\,(1+t^2)}\right) x^{1-C/\log T}\, dt \ll \frac{x^{1-C\log T}}{\varepsilon}. \end{aligned} \quad (14)$$

Now putting together equations (11), (12), (13), (14) we have that

$$\psi_{a,q}(x) + O(\varepsilon x \log x) = \frac{x}{\varphi(q)} + O(\varepsilon x) + O\left(\frac{x}{\varepsilon\sqrt{T}}\right) + O\left(\frac{x^{1-C/\log T}}{\varepsilon}\right).$$

Now we optimize in T by letting $\sqrt{T} = x^{C/\log T}$. Thus, $\log T = \sqrt{2C \log T}$. Now note that

$$x^{-C/\log T} = \exp\left(-\frac{C}{\log T} \log x\right) = \exp\left(-\frac{C}{\sqrt{2C \log x}} \log x\right) = \exp\left(-\sqrt{\frac{C \log x}{2}}\right).$$

So, by absorbing $O(\varepsilon x)$ into $O(\varepsilon x \log x)$ and substituting the above, we have

$$\psi_{a,q}(x) = \frac{x}{\varphi(q)} + O(\varepsilon x \log x) + O\left(\frac{x}{\varepsilon} \exp\left(-\sqrt{\frac{C \log x}{2}}\right)\right).$$

Now we optimize in ε by letting $\varepsilon \log x = \exp\left(-\sqrt{(C \log x)/2}\right)/\varepsilon$. Thus,

$$\varepsilon = \exp\left(-\sqrt{\frac{C \log x}{8}} - \frac{1}{2} \log\log x\right).$$

Now note that

$$\varepsilon x \log x = x \exp\left(-\sqrt{\frac{C \log x}{8}} - \frac{1}{2}\log\log x\right)\exp(\log\log x) = x\exp\left(-\sqrt{\frac{C\log x}{8}} + \frac{1}{2}\log\log x\right).$$

So by substituting the above, we have

$$\psi_{a,q}(x) = \frac{x}{\varphi(q)} + O\left(x\exp\left(-c\sqrt{\log x}\right)\sqrt{\log x}\right)$$

where $c = \sqrt{C/8}$. We can absorb the $\sqrt{\log x}$ into the exponential to get the desired result. □

Corollary 1.51 (Traditional PNT for Arithmetic Progressions)**.** *For $\pi_{a,q}(x)$ the prime counting function of primes less that x equivalent to $a \bmod q$, if $(a,q) = 1$ then we have that*

$$\pi_{a,q}(x) \sim \frac{1}{\phi(q)} \cdot \frac{x}{\log x}$$

Proof. Note that

$$\Pi_{a,q}(x) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a) \sum_{n \le x} \frac{\chi(n)\,\Lambda(n)}{\log n} \qquad \text{and} \qquad \frac{1}{\log n} = \frac{1}{\log x} + \left(-\frac{1}{\log t}\right)\Bigg|_n^x = \frac{1}{\log x} + \int_n^x \frac{dt}{t\log^2 t}.$$

Thus,

$$\Pi_{a,q}(x) = \frac{\psi_{a,q}(x)}{\log x} + \sum_{\substack{n \le x \\ n \equiv a \bmod q}} \Lambda(n) \int_n^x \frac{dt}{t \log^2 t} = \frac{\psi_{a,q}(x)}{\log x} + \int_2^x \frac{\psi_{a,q}(t)\,dt}{t\log^2 t}.$$

Now by the Chebyshev formulation of the PNT, we know that for all $\varepsilon > 0$ there exists x_ε such that for all $x > x_\varepsilon$ we have that $\psi_{a,q}(x) = x/\varphi(q) + O(\varepsilon x)$. Thus,

$$\Pi_{a,q}(x) = \frac{x/\varphi(q) + O(\varepsilon x)}{\log x} + \int_2^x \frac{t/\varphi(q) + O(\varepsilon t)}{t\log^2 t}\,dt = (1/\varphi(q) + O(\varepsilon))\left(\frac{x}{\log x} + \int_2^x \frac{dt}{\log^2 t}\right).$$

Now note that

$$\begin{aligned}\int_2^x \frac{dt}{\log^2 t} &= \int_2^{\sqrt{x}} \frac{dt}{\log^2 t} + \int_{\sqrt{x}}^x \frac{dt}{\log^2 t} \\ &\le \frac{(\sqrt{x}-2)}{\log^2 2} + \frac{(x-\sqrt{x})}{\log^2\sqrt{x}} \le \frac{\sqrt{x}}{\log^2 2} + \frac{4x}{\log^2 x} = O\left(\frac{x}{\log^2 x}\right).\end{aligned}$$

Putting these together and letting $\varepsilon \to 0$ we know that

$$\Pi_{a,q}(x) \sim \frac{1}{\varphi(q)} \cdot \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

Now note that

$$\Pi_{a,q}(x) = \pi_{a,q}(x) + \sum_{\substack{p \le x^{1/2} \\ p^2 \equiv a \bmod q}} \frac{1}{2} + \sum_{\substack{p \le x^{1/3} \\ p^3 \equiv a \bmod q}} \frac{1}{3} + \ldots \le \pi_{a,q}(x) + \frac{x^{1/2}}{2} + \frac{x^{1/3}}{3} + \ldots = \pi_{a,q}(x) + O(\sqrt{x}).$$

So the desired result follows immediately since $\sqrt{x}$ is $O(x/\log^2 x)$. □

Theorem 1.52 (Bombieri-Vinogradov)**.** *For $A \ge 2$ we have*

$$\sum_{q \le Q} \max_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \left|\psi_{a,q}(x) - \frac{x}{\varphi(q)}\right| \ll x(\log x)^{-A}$$

with $Q = \sqrt{x}\,(\log x)^{-B}$ with $B = B(A)$.

Remark. *Note that there are $\sqrt{x}\,(\log x)^{-B}$ terms which sum to something on the order of $x\,(\log x)^{-A}$. So on average, the error term is $x^{1/2+\varepsilon}$. This is almost as good as what we get assuming RH, but this result is an average.*

1.5 Sieve Methods

Lemma 1.53 (Selberg's Inequality)**.** *If λ is a sequence of reals with $\lambda(1) = 1$, then*

$$\sum_{d|n} \mu(d) \leq \left(\sum_{d|n} \lambda(d) \right)^2 .$$

Proof. If $n = 1$ then

$$\sum_{d|n} \mu(d) = \mu(1) = 1 \qquad \text{and} \qquad \left(\sum_{d|n} \lambda(d) \right)^2 = \lambda(1)^2 = 1.$$

Thus the inequality holds. Otherwise, if $n \neq 1$,

$$\sum_{d|n} \mu(d) = 0 \qquad \text{and} \qquad 0 \leq \left(\sum_{d|n} \lambda(d) \right)^2 .$$

Thus the inequality holds. □

Lemma 1.54 (Modular Inversion Formula)**.** *If f and g are functions supported on square-free integers with*

$$g(n) = \sum_{a \equiv 0 \bmod n} f(a),$$

then

$$f(n) = \mu(n) \sum_{a \equiv 0 \bmod n} \mu(a)\, g(a).$$

Proof. We have that

$$\mu(n) \sum_{a \equiv 0 \bmod n} \mu(a)\, g(a) = \mu(n) \sum_{a \equiv 0 \bmod n} \mu(a) \sum_{b \equiv 0 \bmod a} f(b) = \mu(n) \sum_j \mu(jn) \sum_k f(jkn).$$

Now if we let $m = jk$ we reindex the sum

$$\mu(n) \sum_m f(mn) \sum_{j|m} \mu(jn) = \mu^2(n) \sum_m f(mn) \sum_{j|m} \mu(m) = \mu^2(n)\, f(n) = f(n).$$

□

Theorem 1.55 (The Λ^2 Sieve)**.** *Let $\mathcal{A} = (a_n)$ be a sequence of non-negative numbers with support depending on x, and let $\mathcal{P}$ be a finite set of primes with $P = \prod_{p \in \mathcal{P}} p$. For $d \mid P$ we let*

$$A_d(x) = \sum_{n \equiv 0 \bmod d} a_n = w(d)\, x + r(d)$$

where $w(d)$ is a multiplicative function with $0 < w(d) < 1$ and $|r(d)| \leq d \cdot w(d)$ for all $d \mid P$. Then, for an arbitrary choice of $1 < D$, we have

$$S(\mathcal{A}, P) = \sum_{(n,P)=1} a_n \leq \frac{x}{H} + (DH)^2 \qquad \textit{where} \qquad H = \prod_{\substack{p|P \\ p \leq D}} (1 - w(p))^{-1}.$$

Proof. Recall that, by convolution, we have

$$1 = \zeta(s) \cdot \zeta(s)^{-1} = \left(\sum_{1 \leq n} n^{-s} \right) \left(\sum_{1 \leq n} \mu(n)\, n^{-s} \right) \implies \sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by using Lemma 1.53, we have that

$$S(\mathcal{A},P) = \sum_{(n,P)=1} a_n = \sum_n a_n \sum_{d|(n,P)} \mu(d) \le \sum_n a_n \left(\sum_{d|(n,P)} \lambda(d)\right)^2$$
$$= \sum_{d_1|P} \sum_{d_2|P} \lambda(d_1)\,\lambda(d_2) \sum_{n\equiv 0 \bmod [d_1,d_2]} a_n = \sum_{d_1|P} \sum_{d_2|P} \lambda(d_1)\,\lambda(d_2) A_{[d_1,d_2]}(x)$$

where $\lambda(1) = 1$ and λ has some level of support D where $\lambda(d) = 0$ for all $d \ge D$. Now breaking up the A terms

$$S(\mathcal{A},P) = x \sum_{d_1|P} \sum_{d_2|P} \lambda(d_1)\,\lambda(d_2)\, w([d_1,d_2]) + \sum_{d_1|P} \sum_{d_2|P} \lambda(d_1)\,\lambda(d_2)\, r([d_1,d_2]) = x \cdot Q + E.$$

Now note that $(d_1,d_2)\,[d_1,d_2] = d_1 d_2$. So if we let $d_1 = ac$ and $d_2 = bc$ where $c = (d_1,d_2)$, then $abc = [d_1,d_2]$. Using this reindexing we have

$$Q = \sum_c \sum_{ac|P} \sum_{\substack{bc|P\\(a,b)=1}} \lambda(ac)\,\lambda(bc)\,w(abc) = \sum_c w(c) \sum_{ac|P} \sum_{bc|P} \lambda(ac)\,\lambda(bc)\,w(ab) \sum_{d|(a,b)} \mu(d)$$
$$= \sum_{d|P} \mu(d) \sum_c w(c) \sum_{\substack{a\equiv 0 \bmod d\\ b\equiv 0 \bmod d}} \sum \lambda(ac)\,\lambda(bc)\,w(a)\,w(b)$$
$$= \sum_{d|P} \mu(d) \sum_c w(c) \left(\sum_{a\equiv 0 \bmod d} \lambda(ac)\,w(a)\right)^2$$
$$= \sum_{d|P} \mu(d) \sum_c \frac{1}{w(c)} \left(\sum_{a\equiv 0 \bmod d} \lambda(ac)\,w(ac)\right)^2$$
$$= \sum_{d|P} \mu(d) \sum_c \frac{1}{w(c)} \left(\sum_{a\equiv 0 \bmod cd} \lambda(a)\,w(a)\right)^2.$$

So if we let $m = cd$ then by re-indexing, we have

$$Q = \sum_m \left(\sum_{d|(m,P)} \frac{\mu(d)}{w(m/d)}\right) \left(\sum_{a\equiv 0 \bmod m} \lambda(a)\,w(a)\right)^2 = \sum_m x(m)\,y(m)^2$$

where

$$x(m) = \sum_{d|(m,P)} \frac{\mu(d)}{w(m/d)} \qquad \text{and} \qquad y(m) = \sum_{a\equiv 0 \bmod m} \lambda(a)\,w(a)$$

where $y(m)$ also has level of support D where $\lambda(d) = 0$ for all $d \ge D$. Note that, by Lemma 1.54, $y(m)$ is subject to the constraint that

$$\lambda(m)\,w(m) = \mu(m) \sum_{a\equiv 0 \bmod m} \mu(a)\,y(a) = \mu(m) \sum_{\substack{a\le D\\ a\equiv 0 \bmod m}} \mu(a)\,y(a). \tag{15}$$

In the case of $m = 1$, since $\lambda(1) = 1$ we have $1 = w(1) = \sum_a \mu(a)\,y(a)$ (here $1 = w(1)$ comes from the multiplicativity of w). Now by the method of Lagrange multipliers

$$\nabla\left(\sum_m x(m)\,y(m)^2\right) = 2\begin{pmatrix} x(1)\,y(1)\\ x(2)\,y(2)\\ \vdots \end{pmatrix} = \lambda \begin{pmatrix} \mu(1)\\ \mu(2)\\ \vdots \end{pmatrix} = \nabla\left(\sum_a \mu(a)\,y(a)\right).$$

Thus, $y(a) = (\lambda/2)\,(\mu(a)/x(a))$. So we have that

$$1 = w(1) = \sum_a \mu(a)\,y(a) = \sum_{a\le D} \mu(a)\left(\frac{\lambda\,\mu(a)}{2\,x(a)}\right) = \frac{\lambda}{2} \sum_{a\le D} \frac{\mu^2(a)}{x(a)} = \frac{\lambda H}{2} \qquad \text{where} \qquad H = \sum_{\substack{a \text{ square-free}\\ a\le D}} \frac{1}{x(a)}.$$

Thus, $\lambda = 2/H$ and $y(m) = (1/H)(\mu(m)/x(m))$. Substituting this into Equation (15) we find the optimal choice of $\lambda(m)$ is

$$\lambda(m) = \frac{1}{H} \cdot \frac{\mu(m)}{w(m)} \sum_{\substack{a \le D \\ a \equiv 0 \bmod m}} \frac{\mu^2(a)}{x(a)}. \tag{16}$$

Additionally, with this choice of $\lambda(m)$,

$$Q = \sum_m x(m)\, y(m)^2 = \frac{1}{H^2} \sum_{m \le D} \frac{\mu^2(m)}{x(m)} = \frac{1}{H}.$$

Note that

$$x(m) = \sum_{d|(m,P)} \frac{\mu(d)}{w(m/d)} = \frac{1}{w(m)} \sum_{d|m} \mu(d)\, w(d)\, \delta_{d|P} \implies x(p) = \frac{1 - w(p)}{w(p)}.$$

And now since $x(a)^{-1}$ is multiplicative and supported on square-free integers, we have

$$H = \sum_{\substack{a \text{ square-free} \\ a \le D}} \frac{1}{x(a)} = \prod_{\substack{p|P \\ p \le D}} \left(1 + \frac{1}{x(p)}\right) = \prod_{\substack{p|P \\ p \le D}} (1 - w(p))^{-1}.$$

Now, all that remains is to estimate the E term. Returning to Equation (16) and applying the triangle inequality we have

$$|\lambda(m)| \le \frac{1}{H} \cdot \frac{1}{w(m)} \sum_{\substack{a \le D \\ a \text{ square-free} \\ a \equiv 0 \bmod m}} \frac{1}{x(a)} = \frac{1}{H} \cdot \frac{1}{w(m)} \sum_{\substack{am \le D \\ am \text{ square-free}}} \frac{1}{x(am)}$$
$$= \frac{1}{H} \cdot \frac{1}{x(m)\, w(m)} \sum_{\substack{am \le D \\ am \text{ squarefree}}} \frac{1}{x(a)} \le \frac{1}{H} \cdot \frac{1}{x(m)\, w(m)} \sum_{\substack{a \le D \\ a \text{ square-free}}} \frac{1}{x(a)}.$$

Noting that the sum on the RHS is exactly H, we have the inequality

$$|\lambda(m)| \le \frac{1}{x(m)\, w(m)}.$$

Thus,

$$|E| \le \sum_{d_1|P} \sum_{d_2|P} |\lambda(d_1)|\, |\lambda(d_2)|\, |r([d_1, d_2])| \le \sum_{\substack{d_1|P \\ d_1 \le D}} \sum_{\substack{d_2|P \\ d_2 \le D}} \frac{[d_1, d_2]\, w([d_1,\, d_2])}{x(d_1)\, x(d_2)\, w(d_1)\, w(d_2)}.$$

Now note that $w([d_1, d_2])\, w((d_1, d_2)) = w(d_1 d_2)$, and since $0 < w((d_1, d_2)) < 1$ we have that $w([d_1, d_2]) \le w(d_1 d_2)$. Thus,

$$|E| \le \sum_{\substack{d_1|P \\ d_1 \le D}} \sum_{\substack{d_2|P \\ d_2 \le D}} \frac{d_1 d_2\, w(d_1 d_2)}{x(d_1)\, x(d_2)\, w(d_1 d_2)} = \left(\sum_{\substack{d|P \\ d \le D}} \frac{d}{x(d)} \right)^2 \le (DH)^2$$

which completes the proof. □

Definition 1.56 (Almost Prime). *A k-almost prime is a number with at most k prime factors (inclusive).*

Lemma 1.57. *If $n < x$ has no prime factors $p < x^{1/(k+1)}$ then n is a k-almost prime.*

Theorem 1.58 (Brun's Theorem). *We have that*

$$\sum_{p \text{ twin prime}} \frac{1}{p} < \infty.$$

Proof. Let us define a sequence $\mathcal{A}_x = (a_{x,n})$ from the indicator function

$$a_{x,n} = \begin{cases} 1 & n = m\,(m+2) \text{ for some } m < x \\ 0 & \text{otherwise.} \end{cases}$$

and let us define the product

$$P_z = \prod_{p<z} p.$$

Now, if $p < x$ is a twin prime, then $n = p\,(p+2)$ is a 2-almost prime with $a_{x,n} = 1$. Now note that if $z = x^{1/3}$ then if $(n, P_z) = 1$ then n is 2-almost prime by Lemma 1.57. Thus,

$$\pi_2(x) \le S(\mathcal{A}_x, P_z) = \sum_{(n,P_z)=1} a_{x,n}.$$

Now note that

$$A_d(x) = \sum_{n \equiv 0 \bmod d} a_{x,n} = |\{n \equiv 0 \bmod d : n = m\,(m+2),\ m < x\}| = w(d)\,x + r(d)$$

where $w(p) = 1/p$ and $|r(p)| \le 1 = p \cdot w(p)$ when $p = 2$; likewise, $w(p) = 2/p$ and $|r(p)| \le 2 = p \cdot w(p)$ when $p \neq 2$. Thus, we can apply the Λ^2 sieve of Theorem 1.55 to get

$$\pi_2(x) \le S(\mathcal{A}_x, P_z) \le \frac{x}{H} + (DH)^2 \qquad \text{where} \qquad H = \prod_{\substack{p | P_z \\ p \le D}} (1 - w(p))^{-1}.$$

By choosing $D = z$ we give asymptotics on H,

$$H = \prod_{p|P_z} (1 - w(p))^{-1} = 2 \prod_{3 \le p \le z} (1 - 2/p)^{-1} \sim 2 \prod_{3 \le p \le z} (1 - 1/p)^{-2} = \frac{1}{2} \prod_{p \le z} (1 - 1/p)^{-2} \sim 2e^{2\gamma} \log^2 z$$

via Merten's theorem. Thus, we have

$$\pi_2(x) \le S(\mathcal{A}_x, P_z) \ll \frac{x}{\log^2 z} + (z \log^2 z)^2 \ll \frac{x}{\log^2 x} + x^{2/3} \log^4 x \ll \frac{x}{\log^2 x}.$$

Now let $b_n = 1$ if n is a twin prime and 0 otherwise. Then we have that

$$S_N = \sum_{\substack{p \text{ twin prime} \\ p \le N}} \frac{1}{p} = \sum_{n \le N} \frac{b_n}{n} = \frac{1}{N} \sum_{n \le N} b_n - \sum_{n<N} \sum_{k \le n} b_n \left(\frac{1}{n+1} - \frac{1}{n} \right) = \frac{\pi_2(N)}{N} + \sum_{n<N} \frac{\pi_2(n)}{n\,(n+1)}$$

by Abel summation. So applying our sieve estimate we have

$$S_N = \frac{\pi_2(N)}{N} + \sum_{n \le N} \frac{\pi_2(n)}{n\,(n+1)} \ll \frac{1}{\log^2 N} + \sum_{n \le N} \frac{1}{(n+1)\,\log^2 n} \le \frac{1}{\log^2 N} + \sum_{n \le N} \frac{1}{n \log^2 n}.$$

In the limit $N \to \infty$ the leading term vanishes, so S_N behaves however the summation behaves. So we apply the integral test for convergence, with the substitution $u = \log x$

$$\int_2^\infty \frac{dx}{x \log^2 x} = \int_{\log 2}^\infty \frac{du}{u^2} = \left(-\frac{1}{u} \right)_{\log 2}^\infty = \frac{1}{\log 2}.$$

Thus the sum of the reciprocals of twin primes must converge. □

Theorem 1.59. *For any set of δ-spaced points $b_r \in \mathbb{R}/\mathbb{Z}$, and a set of complex numbers a_n with $n \le N$ we have*

$$\sum_r \left| \sum_{n \le N} a_n\, e(b_r n) \right|^2 \le (\delta^{-1} + N) \sum_{n \le N} |a_n|^2.$$

Proof. Too complicated. ☐

Lemma 1.60 (Fixed q w/ Additive Character)**.** *We have that*

$$\sum_{a \bmod q} \left| \sum_{n \le N} a_n \, e(an/q) \right|^2 \le (q+N) \sum_{n \le N} |a_n|^2.$$

Proof. Letting $b_n = n/q$ for n mod q, we note that these are δ-spaced points where $\delta = 1/q$; so this lemma follows as a simple corollary of the above. ☐

Proof. ALTERNATIVE: Note that

$$S = \sum_{a \bmod q} \left| \sum_{n \le N} a_n \, e(an/q) \right|^2 = \sum_{a \bmod q} \left(\sum_{m \le N} a_n \, e(am/q) \right) \left(\sum_{n \le N} \overline{a_n} \, e(-an/q) \right).$$

We rearrange the terms:

$$S = \sum_{m \le N} \sum_{n \le N} a_m \overline{a_n} \sum_{a \bmod q} e(a(m-n)/q).$$

By orthogonality we know this inner sum is q if $n \equiv m$ mod q and is 0 otherwise. So we have

$$S = q \sum_{\substack{m,n \le N \\ m \equiv n \bmod q}}\!\!\sum a_m \overline{a_n} = q \sum_{r \bmod q} \left| \sum_{\substack{n \le N \\ n \equiv r \bmod q}} a_n \right|^2 \le q \sum_{r \bmod q} \left\lceil \frac{N}{q} \right\rceil \sum_{\substack{n \le N \\ n \equiv r \bmod q}} |a_n|^2 \le (q+N) \sum_{n \le N} |a_n|^2$$

by Cauchy-Schwarz. ☐

Lemma 1.61 (Average over $q \le Q$ w/ Additive Character)**.** *We have that*

$$\sum_{q \le Q} \sum_{a \bmod q} \left| \sum_{n \le N} a_n \, e(an/q) \right|^2 \le (Q^2+N) \sum_{n \le N} |a_n|^2.$$

Proof. Note that the set of Farey fractions a/q where $q \le Q$ and $(a,q) = 1$ is δ-spaced where $\delta = Q^{-2}$ because,

$$\left\| \frac{a}{q} - \frac{a'}{q'} \right\| = \left\| \frac{aq' - a'q}{qq'} \right\| \ge \frac{1}{qq'} \ge Q^{-2}.$$

So, this lemma follows as a simple corollary of the δ-spacing lemma. ☐

Lemma 1.62 (Fixed q w/ Multiplicative Character)**.** *We have that*

$$\frac{q}{\varphi(q)} \sum_{\chi \in X_q} \left| \sum_{n \le N} a_n \chi(n) \right|^2 \le (q+N) \sum_{n \le N} |a_n|^2.$$

Proof. Note that

$$S = \sum_{\chi \in X_q} \left| \sum_{n \le N} a_n \chi(n) \right|^2 = \sum_{\chi \in X_q} \left(\sum_{m \le N} a_m \chi(m) \right) \left(\sum_{n \le N} \overline{a_n \chi}(n) \right) = \sum_{m \le N} \sum_{n \le N} a_m \overline{a_n} \sum_{\chi \in X_q} \chi(m) \, \overline{\chi}(n).$$

By orthogonality we then know that

$$S = \varphi(q) \sum_{\substack{m,n \le N \\ m \equiv n \bmod q}}\!\!\sum a_m \overline{a_n} = \varphi(q) \sum_{r \bmod q} \left| \sum_{\substack{n \le N \\ n \equiv r \bmod q}} a_n \right|^2.$$

Thus, recalling the additive case in Lemma 1.60,

$$\frac{q}{\varphi(q)}\, S = q \sum_{r \bmod q} \left| \sum_{\substack{n \le N \\ n \equiv r \bmod q}} a_n \right|^2 = \sum_{a \bmod q} \left| \sum_{n \le N} a_n\, e(an/q) \right|^2 \le (q+N) \sum_{n \le N} |a_n|^2.$$

□

Lemma 1.63 (Average over $q \le Q$ w/ Multiplicative Character)**.** *We have that*

$$\sum_{q \le Q} \frac{q}{\varphi(q)} \sum_{\chi \in X_q} \left| \sum_{n \le N} a_n \chi(n) \right|^2 \le (Q^2 + N) \sum_{n \le N} |a_n|^2.$$

Proof. Note that

$$S = \sum_{\chi \in X_q} \left| \sum_{n \le N} a_n \chi(n) \right|^2 = \sum_{\chi \in X_q} \left(\sum_{m \le N} a_m \chi(m) \right) \left(\sum_{n \le N} \overline{a_n \chi}(n) \right) = \sum_{m \le N} \sum_{n \le N} a_m \overline{a_n} \sum_{\chi \in X_q} \chi(m) \overline{\chi}(n).$$

Now applying orthogonality relations we have that

$$S = \varphi(q) \sum_{\substack{m,n \le N \\ m \equiv n \bmod q}}\sum a_m \overline{a_n} = \varphi(q) \sum_{r \bmod q} \left| \sum_{\substack{n \le N \\ n \equiv r \bmod q}} a_n \right|^2.$$

Recalling the proof of Lemma 1.61, we have that

$$\sum_{q \le Q} \frac{q}{\varphi(q)} S = \sum_{q \le Q} q \sum_{r \bmod q} \left| \sum_{\substack{n \le N \\ n \equiv r \bmod q}} a_n \right|^2 = \sum_{q \le Q} \sum_{a \bmod q} \left| \sum_{n \le N} a_n e(an/q) \right|^2 \le (Q^2 + N) \sum_{n \le N} |a_n|^2.$$

□

Theorem 1.64 (Large-Sieve Type Sum)**.** *If a_m and b_n are sequences defined on $m \le M$ and $n \le N$ with $MN \le x$ with $M, N > x^\delta$ and the b_n satisfy the Seigel-Wielfish condition, then*

$$\sum_{q \le Q} \max_{(a,q)=1} \left| \sum_{mn \equiv a \bmod q}\sum a_m b_n - \frac{1}{\varphi(q)} \sum_{(mn,q)=1}\sum a_m b_n \right| \ll_A x (\log x)^{-A}$$

where $Q = x^{1/2} (\log x)^{-B}$.

Lemma 1.65 (Vaughn's Identity)**.** *We have that*

$$\Lambda(n) = \sum_{\substack{m \mid n \\ m \le z}} \mu(m) \log(n/m) - \sum_{\substack{cm \mid n \\ c \le y,\, m \le z}}\sum \mu(m)\, \Lambda(c) + \sum_{\substack{cm \mid n \\ c > y,\, m > z}}\sum \mu(m)\, \Lambda(c).$$

2 Modular Forms

2.1 Modular Forms and Cusp Forms with Examples

Definition 2.1 (Slash Operator). *For $\gamma \in \mathrm{GL}_2(\mathbb{R})$ we let $f|_k\gamma(z) = \det \gamma^{k/2}\, j(\gamma, z)^{-k}\, f(\gamma z)$.*

Definition 2.2 (Modular Form). *For Γ a subgroup of $\mathrm{SL}_2(\mathbb{Z})$, a weight k modular form $f : \mathbb{C} \to \mathbb{C}$ on Γ is a holomorphic function such that $f(z)$ is bounded as $z \to i\infty$ and satisfies*

$$f(\gamma z) = j(\gamma, z)^k\, f(z) \qquad \text{or equivalently} \qquad f|_k\gamma(z) = f(z)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $\gamma z = (az + b)/(cz + d)$ and $j(\gamma, z) = cz + d$.

Definition 2.3 (Cusp Form). *For Γ a subgroup of $\mathrm{SL}_2(\mathbb{Z})$, a weight k cusp form $f : \mathbb{C} \to \mathbb{C}$ on Γ is a weight k modular form such that $f(\gamma\infty) = 0$ for all $\gamma \in \Gamma$ and has exponential decay as $y \to \infty$.*

Lemma 2.4 (Equivalent Condition on Cusp Forms). *If f is a weight k modular form, then f is a weight k cusp form if and only if $f(\infty) = 0$.*

Proof. ($\Longrightarrow$) Let f be a weight k cusp form. Then $f(\gamma\infty) = 0$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. By modularity we have that

$$f(\infty) = f(\gamma^{-1}\gamma\infty) = j(\gamma^{-1}, \gamma\infty)^k\, f(\gamma\infty) = 0^k \cdot 0 = 0.$$

($\Longleftarrow$) Now suppose that f is a weight k modular form such that $f(\infty) = 0$. Then for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, by modularity we have that

$$f(\gamma\infty) = j(\gamma, \infty)^k\, f(\infty) = (a/c)^k \cdot 0 = 0.$$

Thus, $f(\gamma\infty) = 0$ for all $\gamma \in \Gamma$ and f is a weight k cusp form. □

Lemma 2.5 (Fourier Expansion of Cusp Forms). *If f is a cusp form with Fourier expansion*

$$f(z) = \sum_{0 \le m} a_m\, e(mz)$$

then it must be that $a_0 = 0$.

Proof. Suppose that $a_0 \neq 0$, then $f(i\infty) = a_0 \neq 0$. But by the previous lemma we know this can not be the case, so it must be that $a_0 = 0$. □

Definition 2.6 (Eisenstein Series). *We define the weight k Eisenstein series as*

$$G_k(z) = \sum\sum_{(m,n) \neq (0,0)} (mz + n)^{-k}.$$

We also have the renormalization

$$E_k(z) = \frac{G_k(z)}{2\,\zeta(k)} = \sum\sum_{(m,n)=1} (mz + n)^{-k}.$$

Definition 2.7 (Δ-Function). *Let us define Δ via*

$$\Delta = \frac{E_4^3 - E_6^2}{1728}.$$

Definition 2.8 (Poincare Series). *Let Γ_∞ be the subset of $\Gamma_0(N)$ which are upper triangular. We have that*

$$P_{m,k}(z) = \sum_{\gamma \in \Gamma_\infty / \Gamma_0(N)} e(mz)|_k\gamma = \sum\sum_{\substack{(cN,d)=1 \\ c \ge 0}} \frac{e(m\gamma_{cN,d} z)}{(cNz + d)^k}.$$

Remark. *Note that $P_{0,k}(z) = E_k(z)$ on $\Gamma_0(1) = \Gamma$.*

Lemma 2.9 (Fourier Expansion of Eisenstein Series). *For all k we have that*

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{1 \le m} \sigma_{k-1}(m)\, e(mz).$$

Proof. Recall the Weierstrass factorization for $\sin \pi z$:

$$\sin \pi z = \pi z \prod_{1 \le n} (1 - z/n)(1 + z/n).$$

Taking the logarithm of both sides we have

$$\log(\sin \pi z) = \log(\pi z) + \sum_{1 \le n} \log(1 - z/n) + \log(1 + z/n).$$

Taking the derivative of both sides we have

$$\pi \cot \pi z = \frac{1}{z} + \sum_{1 \le n} \left(\frac{1}{z - n} + \frac{1}{z + n} \right) = \sum_{n \in \mathbb{Z}} \frac{1}{z + n}.$$

But also note that

$$\pi \cot \pi z = \pi i \left(\frac{e(z) + 1}{e(z) - 1} \right) = \pi i - \frac{2\pi i}{1 - e(z)} = \pi i - 2\pi i \sum_{0 \le d} e(dz).$$

Thus,

$$\sum_{n \in \mathbb{Z}} \frac{1}{z + n} = \pi i - 2\pi i \sum_{0 \le d} e(dz) \implies$$

$$(-1)^k \, k! \sum_{n \in \mathbb{Z}} \frac{1}{(z + n)^{k+1}} = \frac{d^k}{dz^k} \sum_{n \in \mathbb{Z}} \frac{1}{z + n} = \frac{d^k}{dz^k} \left(\pi i - 2\pi i \sum_{0 \le d} e(dz) \right) = -2\pi i \, (2\pi i)^k \sum_{0 \le d} d^k \, e(dz).$$

And so, via algebraic manipulation, we have that

$$\sum_{n \in \mathbb{Z}} \frac{1}{(z + n)^{k+1}} = \frac{(-2\pi i)^{k+1}}{k!} \sum_{0 \le d} d^k \, e(dz).$$

Now note that

$$G_k(z) = \sum_{(m,n) \neq (0,0)} \sum (mz + n)^{-k} = 2\,\zeta(k) + 2 \sum_{1 \le m} \sum_{n \in \mathbb{Z}} (mz + n)^{-k} = 2\,\zeta(k) + 2 \left(\frac{(-2\pi i)^k}{(k - 1)!} \right) \sum_{1 \le m} \sum_{0 \le d} d^{k-1} \, e(dmz).$$

Now since $\sigma_k(n) = \sum_{d|n} d^k$ we have that

$$G_k(z) = 2\,\zeta(k) + 2 \left(\frac{(-2\pi i)^k}{\Gamma(k)} \right) \sum_{1 \le n} \sigma_k(n) \, e(nz).$$

Dividing through by $2\,\zeta(k)$ we have that

$$E_k(z) = 1 + \frac{(-2\pi i)^k}{\zeta(k)\,\Gamma(k)} \sum_{1 \le n} \sigma_{k-1}(n) \, e(nz).$$

Since $(-2\pi i)^k / (\zeta(k)\,\Gamma(k)) = -2k/B_k$ the result immediately follows. $\square$

Definition 2.10 (Fourier Expansion of Δ-Function)**.** *We define τ as the coefficients of the Fourier expansion of Δ:*

$$\Delta(z) = \sum_{1 \le m} \tau(m) \, e(mz).$$

Note that since Δ is a weight 12 cusp form, we have that $\tau(0) = 0$.

Lemma 2.11 (Fourier Expansion of Poincare Series)**.** *For $m > 0$ we have that*

$$P_{m,k}(z) = \sum_{0 \le n} \left(2\delta_{m,n} + 2\pi i^k \left(\frac{n}{m} \right)^{(k-1)/2} \sum_{1 \le c} \frac{S(m, n, cN)}{cN} \, J_{k-1} \left(\frac{4\pi \sqrt{mn}}{cN} \right) \right) e^{2\pi i n z}.$$

Proof. Note that if

$$P_{m,k}(z) = \sum_{0\le n} a_n e^{2\pi i n z}$$

then we have that

$$a_n = \int_{0+iy}^{1+iy} P_{m,k}(z)\, e(-nz)\, dz.$$

Substituting our definition of $P_{m,k}(z)$ we have that

$$a_n = \sum_{\substack{(cN,d)=1\\ c\ge 0}}\sum \int_{0+iy}^{1+iy} \frac{e(m\gamma_{cN,d}z)}{(cNz+d)^k}\, e(-nz)\, dz.$$

Now focusing on the $c=0$ terms we have

$$\sum_{d=\pm 1}\int_{0+iy}^{1+iy} e(mz)\, e(-nz)\, dz = 2\delta_{m,n}.$$

Thus, breaking our remaining sum over d into equivalence classes we have

$$a_n = 2\delta_{m,n} + \sum_{1\le c}\sum_{d\in(\mathbb{Z}/cN\mathbb{Z})^\times}\sum_{l\in\mathbb{Z}}\int_{0+iy}^{1+iy} \frac{e(m\gamma_{cN,d+cNl}z)}{(cN(z+l)+d)^k} e(-nz)\, dz.$$

Now noting that $a\equiv d^{-1}\equiv (d+cNl)^{-1} \bmod cN$, we have that

$$\gamma_{c,d+cNl}z = \frac{az + (a(d+cNl)-1)/(cN)}{cNz+(d+cNl)} = \frac{a(z+l)+(ad-1)/(cN)}{cN(z+l)+d} = \gamma_{cN,d}(z+l).$$

So applying this and the substitution $z\mapsto z-l$ we have

$$\begin{aligned} a_n &= 2\delta_{m,n} + \sum_{1\le c}\sum_{d\in(\mathbb{Z}/cN\mathbb{Z})^\times}\sum_{l\in\mathbb{Z}}\int_{l+iy}^{l+1+iy} \frac{e(m\gamma_{cN,d}z)}{(cNz+d)^k}\, e(-nz)\, dz \\ &= 2\delta_{m,n} + \sum_{1\le c}\sum_{d\in(\mathbb{Z}/cN\mathbb{Z})^\times}\int_{-\infty+iy}^{\infty+iy} \frac{e(m\gamma_{cN,d}z)}{(cNz+d)^k}\, e(-nz)\, dz \end{aligned}$$

Now note that

$$\gamma_{cN,d}z = \frac{az+(ad-1)/(cN)}{cNz+d} = \frac{a}{cN} - \frac{1}{(cN)(cNz+d)};$$

thus,

$$a_n = 2\delta_{m,n} + \sum_{1\le c}\sum_{d\in(\mathbb{Z}/cN\mathbb{Z})^\times}\int_{-\infty+iy}^{\infty+iy} (cNz+d)^{-k}\, e\left(\frac{ma}{cN} - \frac{m}{(cN)(cNz+d)}\right)\, e(-nz)\, dz.$$

Now substituting $z\mapsto z-d/(cN)$ we have

$$\begin{aligned} a_n &= 2\delta_{m,n} + \sum_{1\le c}\sum_{d\in(\mathbb{Z}/cN\mathbb{Z})^\times}\int_{-\infty+iy}^{\infty+iy} (cNz)^{-k}\, e\left(\frac{ma}{cN} - \frac{m}{(cN)^2 z}\right)\, e\left(-nz+\frac{nd}{cN}\right)\, dz \\ &= 2\delta_{m,n} + \sum_{1\le c}\sum_{d\in(\mathbb{Z}/cN\mathbb{Z})^\times} e\left(\frac{ma+nd}{cN}\right)\int_{-\infty+iy}^{\infty+iy} (cNz)^{-k} e\left(-\frac{m}{(cN)^2 z} - nz\right)\, dz. \end{aligned}$$

Now note the Kloosterman sum in the formula, and making the substitution $z\mapsto -\sqrt{m/n}\,(z/c)$ we have

$$a_n = 2\delta_{m,n} + \left(\frac{n}{m}\right)^{k/2}\sum_{1\le c} S(m,n,cN)\int_{-\infty+iy}^{\infty+iy} z^{-k} e\left(\frac{\sqrt{mn}}{cN}(z+z^{-1})\right)\, dz.$$

This last integeral can be massaged into the appropriate J-Bessel function (and scaling factors) with some manipulation. □

2.2 The Space of M_k and S_k

Theorem 2.12 (Valence Formula)**.** *Let $v_p(f)$ be the order of vanishing a non-zero function f at a point $z = p$. Then if f is a weight k modular form we have that*

$$\sum_{z \in \mathcal{H}/\mathrm{SL}_2(\mathbb{Z})} \frac{v_z(f)}{w_z} = \frac{k}{12}$$

where $w_z = 2$ if $z = i$, $w_z = 3$ if $z = \zeta_6$, and $w_z = 1$ otherwise.

Proof. (Sketch) Apply the argument principle to the contour going around the fundamental domain with cutouts around the points at i, ζ_6, ζ_3, and ∞. □

Lemma 2.13 (Non-Vanishing of Δ)**.** *We have that Δ is non-vanishing on $\mathcal{H}$ except for a simple zero at $i\infty$.*

Proof. Note that Δ is of weight 12 and $\Delta(i\infty) = 0$ via the Fourier expansion coming from

$$\Delta = \frac{E_4^3 - E_6^2}{1728}.$$

Thus we know that $v_{i\infty}(\Delta) \geq 1$ and via the Valence formula we have

$$v_{i\infty}(\Delta) + \sum_{z \in (\mathcal{H}/\mathrm{SL}_2(\mathbb{Z})) \setminus \{i\infty\}} \frac{v_z(\Delta)}{w_z} = 1.$$

Since Δ is holomorphic, we know that $v_z(\Delta) \geq 0$ for all z; thus it must be the case that $v_{i\infty}(\Delta) = 1$ and $v_z(\Delta) = 0$ for all $z \neq i\infty$ as desired. □

Lemma 2.14 (Preliminary to Characterizing the Space M_k I)**.** *For $k < 0$ there are no modular forms of weight k.*

Proof. Suppose that f is a modular form of weight $k < 0$. Then by the valence formula we know that there exists $z \in \mathcal{H}/\mathrm{SL}_2(\mathbb{Z})$ such that $v_z(f) < 0$. But this would imply f is non-holomorphic, contradiction. □

Lemma 2.15 (Preliminary to Characterizing the Space M_k II)**.** *We have that $M_k = \Delta \cdot M_{k-12} \oplus \mathbb{C} \cdot E_k$.*

Proof. Let $f \in M_k$ and let $a_0 = f(i\infty)$. Now note that $f - a_0 E_k$ has a zero at $i\infty$; thus, since Δ is non-vanishing on $\mathcal{H}$ and has a simple zero at $i\infty$ (per Lemma 2.13), we have that $(f - a_0 E_k)/\Delta = g$ is a weight $k - 12$ modular form. Thus,

$$f = \Delta \cdot g + a_0 E_k$$

and the result immediately follows. □

Theorem 2.16 (Characterization of the Space M_k)**.** *We have the following results on the space M_k.*

- *If k is odd then $M_k = 0$.*
- *If $k = 0$ then $M_k = \mathbb{C}$.*
- *If $k = 2$, then $M_k = 0$.*
- *If $4 \leq k \leq 10$ is even, then $M_k = \mathbb{C} \cdot E_k$.*
- *If $k \geq 12$, then $M_k = \Delta \cdot M_{k-12} \oplus \mathbb{C} \cdot E_k$.*

Proof. We cover each of the six cases above:

- Note that if there exists a non-zero modular form f of odd weight k on $\mathrm{SL}_2(\mathbb{Z})$, then for $\gamma = -I_2$ we have that

  $$f(z) = f(\gamma z) = (-1)^k f(z) = -f(z).$$

  This is a contradiction because it implies that f vanishes everywhere.

- If $k = 0$ then $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$. For fixed z, since the set $\{\gamma z : \gamma \in \Gamma\}$ has accumulation points, we know that $f(z)$ is constant by the identity theorem.

- If $k = 2$ then there is no integral solution to the valence formula because any vanishing point that might exist contributes at least $+1/3$. Thus $M_2 = 0$.
- If $4 \le k \le 10$ is even, then $M_k = \Delta \cdot M_{k-12} \oplus \mathbb{C} \cdot E_k$ with $k - 12 < 0$. So by Lemma 2.14, $M_k = \mathbb{C} \cdot E_k$.
- This is exactly Lemma 2.15.

□

Corollary 2.17 (Dimension of M_k). *For $k \ge 0$, we have that*

$$\dim_{\mathbb{C}} M_k = \begin{cases} \left\lfloor \dfrac{k}{12} \right\rfloor + 1 & k \not\equiv 2 \bmod 12 \\ \left\lfloor \dfrac{k}{12} \right\rfloor & k \equiv 2 \bmod 12 \end{cases}$$

Proof. Trivial. It immediately follows via downwards induction. □

Theorem 2.18 (E_4 and E_6 Generate M). *We have $M = \mathbb{C}[E_4, E_6]$ where M is the space of modular forms.*

Proof. Trivially, $M_0, M_2, M_4, M_6 \subseteq \mathbb{C}[E_4, E_6]$ by Theorem 2.16. Now note that E_4^2 is a weight 8 modular form, so by Theorem 2.16 we know that $E_4^2 = c\, E_8$ for some non-zero $c \in \mathbb{C}$. Thus, $M_8 \subseteq \mathbb{C}[E_4, E_6]$. Likewise, since $E_4 E_6$ is a weight 10 modular form, by Theorem 2.16 we know that $E_4 E_6 = c\, E_{10}$ for some non-zero $c \in \mathbb{C}$. Thus, $M_{10} \subseteq \mathbb{C}[E_4, E_6]$.

Now if we suppose that $M_k \subseteq \mathbb{C}[E_4, E_6]$, let us show that $M_{k+12} \subseteq \mathbb{C}[E_4, E_6]$. Note there exists a, b such that $E_4^a E_6^b$ is a weight $k + 12$ modular form, so by Theorem 2.16 we know that

$$E_4^a E_6^b = \Delta \cdot f + c \cdot E_{k+12}$$

for some $c \in \mathbb{C}$ and $f \in M_k$. Note that $c \ne 0$ since $E_4^a E_6^b \notin S_{k+12}$; so, since $f \in M_k \subseteq \mathbb{C}[E_4, E_6]$ we know that

$$E_{k+12} = c^{-1} \left(E_4^a E_6^b - \Delta \cdot f \right)$$

is in $\mathbb{C}[E_4, E_6]$. Since, $M_{k+12} = \Delta \cdot M_k \oplus \mathbb{C} \cdot E_{k+12}$ with $M_k \subseteq \mathbb{C}[E_4, E_6]$ and $E_{k+12} \in \mathbb{C}[E_4, E_6]$, we know that $M_{k+12} \subseteq \mathbb{C}[E_4, E_6]$ as desired. Now applying induction we have that $M_k \subseteq \mathbb{C}[E_4, E_6]$ for all even k. Since

$$M = \bigcup_k M_{2k}$$

as per Theorem 2.16, we know that $M \subseteq \mathbb{C}[E_4, E_6]$. The reverse inclusion is trivial and the result follows. □

Theorem 2.19 (Characterization of the Space S_k). *We have that $S_k = \Delta \cdot M_{k-12}$.*

Proof. Let $\pi : M_{k-12} \to S_k$ via $f \mapsto \Delta \cdot f$. Note that if $g \in S_k$ then $g/\Delta \in M_{k-12}$ because Δ is non-vanishing on $\mathcal{H}$ and $v_{i\infty}(g) \ge 1$ with $v_{i\infty}(\Delta) = 1$. So π is surjective. Additionally, note that if $\Delta \cdot f = 0$ then $f = 0$ since Δ is non-vanishing on $\mathcal{H}$. So π is injective.

Thus, π is bijective and the result immediately follows. □

Corollary 2.20 (Dimension of S_k). *For $k < 12$ we have that $\dim_{\mathbb{C}} S_k = 0$. Otherwise, $\dim_{\mathbb{C}} S_k = \dim_{\mathbb{C}} M_k - 1$.*

Proof. For $k < 12$ we know that $S_k = \Delta \cdot M_{k-12}$ with $k - 12 < 0$ via Theorem 2.19, and we know that M_{k-12} is empty via Lemma 2.14; thus, S_k is empty for $k < 12$. So $\dim_{\mathbb{C}} S_k = 0$.

For $k \ge 12$ we know that $S_k = \Delta \cdot M_{k-12}$ with $k - 12 \ge 0$ via Theorem 2.19, and we know that $M_k = \Delta \cdot M_{k-12} \oplus \mathbb{C} \cdot E_k$ via Lemma 2.15. Thus, $\dim_{\mathbb{C}} S_k = \dim_{\mathbb{C}} M_{k-12}$ and $\dim_{\mathbb{C}} M_k = \dim_{\mathbb{C}} M_{k-12} + 1$, by algebraic manipulation the result immediately follows. □

Lemma 2.21 (Arithmetic Identity on σ_7 from $E_4^2 = E_8$). *We have that*

$$\sigma_7(m) = \sigma_3(m) + 120 \sum_{0<n<m} \sigma_3(n)\, \sigma_3(m-n).$$

Proof. From Corollary 2.17 we know that $\dim_{\mathbb{C}} M_8 = 1$. Thus, $E_4^2 = E_8$ up to constant multiple. Of course, Lemma 2.9 gives us that the Fourier expansion of E_4 and E_8 both have constant term 1. So, $E_4^2 = E_8$ exactly.

From Lemma 2.9 we know that

$$E_4 = 1 - \frac{8}{B_4} \sum_{1 \leq m} \sigma_3(m)\, e(mz) \qquad \text{and} \qquad E_8 = 1 - \frac{16}{B_8} \sum_{1 \leq m} \sigma_7(m)\, e(mz).$$

Now we compute

$$E_4^2 = \left(1 - \frac{8}{B_4} \sum_{1 \leq m} \sigma_3(m)\, e(mz)\right)^2 = 1 - \frac{16}{B_4} \sum_{1 \leq m} \sigma_3(m)\, e(mz) + \frac{64}{B_4^2} \left(\sum_{1 \leq m} \sigma_3(m)\, e(mz)\right)^2.$$

Now we use convolution to square the summation on the right

$$E_4^2 = 1 + \sum_{1 \leq m} \left(-\frac{16}{B_4}\, \sigma_3(m) + \frac{64}{B_4^2} \sum_{0<n<m} \sigma_3(n)\, \sigma_3(m-n)\right) e(mz).$$

Now we match Fourier coefficients to get

$$-\frac{16}{B_8}\, \sigma_7(m) = -\frac{16}{B_4}\, \sigma_3(m) + \frac{64}{B_4^2} \sum_{0<n<m} \sigma_3(n)\, \sigma_3(m-n).$$

Recalling that $B_4 = -1/30$ and $B_8 = -1/30$ we get

$$\sigma_7(m) = \sigma_3(m) + 120 \sum_{0<n<m} \sigma_3(n)\, \sigma_3(m-n).$$

□

Lemma 2.22 (Arithmetic Identity on σ_9 from $E_4 E_6 = E_{10}$)**.** *We have that*

$$\sigma_9(m) = -\frac{10}{11}\, \sigma_3(m) + \frac{21}{11}\, \sigma_5(m) + \frac{5040}{11} \sum_{0<n<m} \sigma_3(n)\, \sigma_5(m-n).$$

Proof. From Corollary 2.17 we know that $\dim_{\mathbb{C}} M_{10} = 1$. Thus, $E_4 E_6 = E_{10}$ up to constant multiple. Of course, Lemma 2.9 gives us that the Fourier expansion of E_4, E_6, and E_{10} all have constant term 1. So $E_4 E_6 = E_{10}$ exactly.

From Lemma 2.9 we know that

$$E_4 = 1 - \frac{8}{B_4} \sum_{1 \leq m} \sigma_3(m)\, e(mz) \qquad \text{and} \qquad E_6 = 1 - \frac{12}{B_6} \sum_{1 \leq m} \sigma_5(m)\, e(mz).$$

Now we compute

$$\begin{aligned} E_4 E_6 &= \left(1 - \frac{8}{B_4} \sum_{1 \leq m} \sigma_3(m)\, e(mz)\right) \left(1 - \frac{12}{B_6} \sum_{1 \leq m} \sigma_5(m)\, e(mz)\right) \\ &= 1 - \frac{8}{B_4} \sum_{1 \leq m} \sigma_3(m)\, e(mz) - \frac{12}{B_6} \sum_{1 \leq m} \sigma_5(m)\, e(mz) + \frac{96}{B_4 B_6} \left(\sum_{1 \leq m} \sigma_3(m)\, e(mz)\right) \left(\sum_{1 \leq m} \sigma_5(m)\, e(mz)\right). \end{aligned}$$

Now we use convolution to evaluate the product of sums on the right

$$E_4 E_6 = 1 + \sum_{1 \leq m} \left(-\frac{8}{B_4}\, \sigma_3(m) - \frac{12}{B_6}\, \sigma_5(m) + \frac{96}{B_4 B_6} \sum_{0<n<m} \sigma_3(n)\, \sigma_5(m-n)\right) e(mz).$$

Now we match Fourier coefficients to get

$$-\frac{20}{B_{10}}\, \sigma_9(m) = -\frac{8}{B_4}\, \sigma_3(m) - \frac{12}{B_6}\, \sigma_5(m) + \frac{96}{B_4 B_6} \sum_{0<n<m} \sigma_3(n)\, \sigma_5(m-n).$$

Recalling that $B_4 = -1/30$, $B_6 = 1/42$, and $B_{10} = 5/66$ we get

$$\sigma_9(m) = -\frac{10}{11}\,\sigma_3(m) + \frac{21}{11}\,\sigma_5(m) + \frac{5040}{11}\sum_{0<n<m} \sigma_3(n)\,\sigma_5(m-n).$$

□

Lemma 2.23 (Arithmetic Identity on σ_{13} from $E_4E_{10} = E_{14}$)**.** *We have that*

$$\sigma_{13}(m) = -10\,\sigma_3(m) + 11\,\sigma_9(m) + 2640\sum_{0<n<m} \sigma_3(n)\,\sigma_9(m-n).$$

Proof. From Corollary 2.17 we know that $\dim_{\mathbb{C}} M_{14} = 1$. Thus, $E_4E_{10} = E_{14}$ up to constant multiple. Of course, Lemma 2.9 gives us that the Fourier expansion of E_4, E_{10}, and E_{14} all have constant term 1. So $E_4E_{10} = E_{14}$ exactly.

From Lemma 2.9 we know that

$$E_4 = 1 - \frac{8}{B_4}\sum_{1\le m} \sigma_3(m)\,e(mz) \qquad \text{and} \qquad E_{10} = 1 - \frac{20}{B_{10}}\sum_{1\le m} \sigma_9(m)\,e(mz).$$

Now we compute

$$\begin{aligned} E_4E_{10} &= \left(1 - \frac{8}{B_4}\sum_{1\le m} \sigma_3(m)\,e(mz)\right)\left(1 - \frac{20}{B_{10}}\sum_{1\le m} \sigma_9(m)\,e(mz)\right) \\ &= 1 - \frac{8}{B_4}\sum_{1\le m} \sigma_3(m)\,e(mz) - \frac{20}{B_{10}}\sum_{1\le m} \sigma_9(m)\,e(mz) + \frac{160}{B_4B_{10}}\left(\sum_{1\le m} \sigma_3(m)\,e(mz)\right)\left(\sum_{1\le m} \sigma_9(m)\,e(mz)\right). \end{aligned}$$

Now we use convolution to evaluate the product of sums on the right

$$E_4E_{10} = 1 + \sum_{1\le m}\left(-\frac{8}{B_4}\,\sigma_3(m) - \frac{20}{B_{10}}\,\sigma_9(m) + \frac{160}{B_4B_{10}}\sum_{0<n<m} \sigma_3(n)\,\sigma_9(m-n)\right) e(mz).$$

Now we match Fourier coefficients to get

$$-\frac{28}{B_{14}}\,\sigma_{13}(m) = -\frac{8}{B_4}\,\sigma_3(m) - \frac{20}{B_{10}}\,\sigma_9(m) + \frac{160}{B_4B_{10}}\sum_{0<n<m} \sigma_3(n)\,\sigma_9(m-n).$$

Recalling that $B_4 = -1/30$, $B_{10} = 5/66$, and $B_{14} = 7/6$ we get

$$\sigma_{13}(m) = -10\,\sigma_3(m) + 11\,\sigma_9(m) + 2640\sum_{0<n<m} \sigma_3(n)\,\sigma_9(m-n).$$

□

Lemma 2.24 (Arithmetic Identity on σ_{13} from $E_6E_8 = E_{14}$)**.** *We have that*

$$\sigma_{13}(m) = 21\,\sigma_5(m) - 20\,\sigma_7(m) + 10080\sum_{0<n<m} \sigma_5(n)\,\sigma_7(m-n).$$

Proof. From Corollary 2.17 we know that $\dim_{\mathbb{C}} M_{14} = 1$. Thus $E_6E_8 = E_{14}$ up to constant multiple. Of course, Lemma 2.9 gives us that the Fourier expansion of E_6, E_8, and E_{14} all have constant term 1. So $E_6E_8 = E_{14}$ exactly.

From Lemma 2.9 we know that

$$E_6 = 1 - \frac{12}{B_6}\sum_{1\le m} \sigma_5(m)\,e(mz) \qquad \text{and} \qquad E_8 = 1 - \frac{16}{B_8}\sum_{1\le m} \sigma_7(m)\,e(mz).$$

Now we compute

$$E_6 E_8 = \left(1 - \frac{12}{B_6}\sum_{1\le m}\sigma_5(m)\,e(mz)\right)\left(1 - \frac{16}{B_8}\sum_{1\le m}\sigma_7(m)\,e(mz)\right)$$
$$= 1 - \frac{12}{B_6}\sum_{1\le m}\sigma_5(m)\,e(mz) - \frac{16}{B_8}\sum_{1\le m}\sigma_7(m)\,e(mz) + \frac{192}{B_6 B_8}\left(\sum_{1\le m}\sigma_5(m)\,e(mz)\right)\left(\sum_{1\le m}\sigma_7(m)\,e(mz)\right).$$

Now we use the convolution to evaluate the product of sums on the right

$$E_6 E_8 = 1 + \sum_{1\le m}\left(-\frac{12}{B_6}\,\sigma_5(m) - \frac{16}{B_8}\,\sigma_7(m) + \frac{192}{B_6 B_8}\sum_{0<n<m}\sigma_5(n)\,\sigma_7(m-n)\right)e(mz).$$

Now we match Fourier coefficients to get

$$-\frac{28}{B_{14}}\,\sigma_{13}(m) = -\frac{12}{B_6}\,\sigma_5(m) - \frac{16}{B_8}\,\sigma_7(m) + \frac{192}{B_6 B_8}\sum_{0<n<m}\sigma_5(n)\,\sigma_7(m-n).$$

Recalling that $B_6 = 1/42$, $B_8 = -1/30$, and $B_{14} = 7/6$ we get

$$\sigma_{13}(m) = 21\,\sigma_5(m) - 20\,\sigma_7(m) + 10080\sum_{0<n<m}\sigma_5(n)\,\sigma_7(m-n).$$

□

Lemma 2.25 (Computing τ from $E_6^2 - E_{12} = a\,\Delta$)**.** *We have that*

$$\tau(m) = \frac{65}{756}\,\sigma_{11}(m) + \frac{691}{756}\,\sigma_5(m) - \frac{691}{3}\sum_{0<n<m}\sigma_5(n)\,\sigma_5(m-n).$$

Proof. Recall from Lemma 2.16 that $M_{12} = \Delta \cdot M_0 \oplus \mathbb{C}\cdot E_{12}$ and $M_0 = \mathbb{C}$. Thus, $M_{12} = \Delta\cdot\mathbb{C} \oplus \mathbb{C}\cdot E_{12}$. Thus, since E_6^2 is a weight 12 modular form, we have that $E_6^2 = a\,\Delta + b\,E_{12}$ for some $a, b \in \mathbb{C}$. Of course, Lemma 2.9 gives us that the Fourier expansion of E_6 and E_{12} have constant term 1, and we know that the constant term of a cusp form is 0. Thus, $b = 1$ and we have that $E_6^2 - E_{12} = a\,\Delta$.

From Lemma 2.9 we know that

$$E_6 = 1 - \frac{12}{B_6}\sum_{1\le m}\sigma_5(m)\,e(mz) \qquad \text{and} \qquad E_{12} = 1 - \frac{24}{B_{12}}\sum_{1\le m}\sigma_{11}(m)\,e(mz).$$

Now we compute

$$E_6^2 = \left(1 - \frac{12}{B_6}\sum_{1\le m}\sigma_5(m)\,e(mz)\right)^2 = 1 - \frac{24}{B_6}\sum_{1\le m}\sigma_5(m)\,e(mz) + \frac{144}{B_6^2}\left(\sum_{1\le m}\sigma_5(m)\,e(mz)\right)^2.$$

Now we use convolution to square the summation on the right

$$E_6^2 = 1 + \sum_{1\le m}\left(-\frac{24}{B_6}\,\sigma_5(m) + \frac{144}{B_6^2}\sum_{0<n<m}\sigma_5(n)\,\sigma_5(m-n)\right)e(mz).$$

Now we subtract off E_{12} to get

$$E_6^2 - E_{12} = \sum_{1\le m}\left(\frac{24}{B_{12}}\,\sigma_{11}(m) - \frac{24}{B_6}\,\sigma_5(m) + \frac{144}{B_6^2}\sum_{0<n<m}\sigma_5(n)\,\sigma_5(m-n)\right)e(mz).$$

Now we match the $m = 1$ Fourier coefficient to get

$$a = a\,\tau(1) = \frac{24}{B_{12}}\,\sigma_{11}(1) - \frac{24}{B_6}\,\sigma_5(1) + \frac{144}{B_6^2}\sum_{0<n<1}\sigma_5(n)\,\sigma_5(1-n) = \frac{24}{B_{12}} - \frac{24}{B_6}.$$

Now we match Fourier coefficients to get

$$\left(\frac{24}{B_{12}} - \frac{24}{B_6}\right) \tau(m) = \frac{24}{B_{12}}\, \sigma_{11}(m) - \frac{24}{B_6}\, \sigma_5(m) + \frac{144}{B_6^2} \sum_{0<n<m} \sigma_5(n)\, \sigma_5(m-n).$$

Recalling that $B_6 = 1/42$ and $B_{12} = -691/2730$ we get

$$\tau(m) = \frac{65}{756}\, \sigma_{11}(m) + \frac{691}{756}\, \sigma_5(m) - \frac{691}{3} \sum_{0<n<m} \sigma_5(n)\, \sigma_5(m-n).$$

□

2.3 L-Functions Associated with Cusp Forms

Definition 2.26 (L-Functions Associated with Cusp Forms)**.** *Given a cusp form $f \in S_k$ with Fourier expansion*

$$f(z) = \sum_{1 \leq n} a_n e^{2\pi i n z}$$

we define the L-function associated with this modular form as $L(f,s)$ where

$$L(f,s) = \sum_{1 \leq n} \frac{a_n}{n^s}.$$

Lemma 2.27 (Hecke Bound for Cusp Forms)**.** *Given a cusp form $f \in S_k$ with Fourier expansion*

$$f(z) = \sum_{1 \leq n} a_n e^{2\pi i n z}$$

we have that $|a_n| \ll n^{k/2}$.

Proof. Since $f \in S_k$ we know that f has exponential decay as $y \to \infty$ and that f is Γ-periodic. Thus the function $y^{k/2}|f|$ is bounded on $\mathcal{H}$, and we have that $|f| \ll y^{-k/2}$.

Noting that

$$f(z) = \sum_{1 \leq n} a_n e^{2\pi i n z} \qquad \text{and} \qquad \overline{f(z)} = \sum_{1 \leq n} \overline{a_n} e^{-2\pi i n \overline{z}}$$

by Parseval's identity we know that

$$\sum_{n \in \mathbb{Z}} |a_n|^2 e^{-4\pi n y} = \sum_{n \in \mathbb{Z}} \left(a_n e^{2\pi i n z}\right)\left(\overline{a_n} e^{-2\pi i n \overline{z}}\right) = \int_0^1 |f(z)|^2 \, dz \ll y^{-k}.$$

So we have that

$$\sum_{n \leq N} |a_n|^2 \leq e^{4\pi N y} \sum_{n \leq N} |a_n|^2 e^{-4\pi n y} \leq e^{4\pi N y} \sum_{n \in \mathbb{Z}} |a_n|^2 e^{-4\pi n y} \ll y^{-k} e^{4\pi N y}$$

for all $y > 0$. Choosing $y = 1/N$ we have that

$$|a_N|^2 \leq \sum_{n \leq N} |a_n|^2 \ll (1/N)^{-k} e^{4\pi N\,(1/N)} = e^{4\pi} N^k \ll N^k.$$

Taking the square root of both sides yields the desired result. ◻

Lemma 2.28. *If $f \in S_k$, then We have that $L(f,s)$ is absolutely convergent for $\sigma > 1 + k/2$.*

Proof. If $\sigma > 1 + k/2$ then we know that $|a_n/n^s| \ll n^{-(\sigma - k/2)}$ with $\sigma - k/2 > 1$ by the Hecke bound. Thus we have absolute convergence of $L(f,s)$ and $|L(f,s)| \ll \zeta(\sigma - k/2)$ by the triangle inequality. ◻

Theorem 2.29 (Functional Equation for L-Functions Associated with Cusp Forms)**.** *For $f \in S_k$, we can extend $L(f,s)$ to an entire function, and if*

$$\Lambda(f,s) = \mathcal{M}(f(it))(s) = (2\pi)^{-s}\,\Gamma(s)\,L(f,s)$$

then we have that

$$\Lambda(f,s) = i^k\,\Lambda(f, k-s).$$

Proof. Recall that for all $\sigma > 0$, $\Gamma(s)$ is defined as

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt.$$

Now if we do the substitution $t = 2\pi n u$, then after some manipulation we have that

$$(2\pi n)^{-s}\,\Gamma(s) = \int_0^\infty e^{-2\pi n t} t^{s-1}\, dt.$$

Now if we take the sum over all $n \geq 1$ with the coefficients a_n.

$$(2\pi)^{-s} L(f,s)\,\Gamma(s) = \sum_{1 \leq n} a_n\,(2\pi n)^{-s}\,\Gamma(s) = \sum_{1 \leq n} a_n \int_0^\infty e^{-2\pi n t} t^{s-1}\,dt.$$

Now note that, using Hecke's bound and the substitution $u = 2\pi n t$ we have

$$\begin{aligned}\sum_{1 \leq n} \int_0^\infty \left|a_n\,e^{-2\pi n t} t^{s-1}\right| dt &\ll \sum_{1 \leq n} n^{k/2} \int_0^\infty e^{-2\pi n t} t^{\sigma-1}\,dt \\ &= (2\pi)^{-\sigma} \sum_{1 \leq n} n^{-(\sigma-k/2)} \int_0^\infty e^{-u} u^{\sigma-1}\,du = \frac{\zeta(\sigma - k/2)\,\Gamma(\sigma)}{(2\pi)^{\sigma}}.\end{aligned}$$

For $\sigma > 1 + k/2$ we know this converges. So by Fubini-Tonelli, when $\sigma > 1 + k/2$, we can interchange the sum and the integral as needed.

$$(2\pi)^{-s} L(f,s)\,\Gamma(s) = \sum_{1 \leq n} a_n \int_0^\infty e^{-2\pi n t} t^{s-1}\,dt = \int_0^\infty \sum_{1 \leq n} a_n e^{-2\pi n t}\,t^{s-1}\,dt = \int_0^\infty f(it)\,t^{s-1}\,dt = \mathcal{M}(f(it))(s).$$

Now note that

$$(2\pi)^{-s} L(f,s)\,\Gamma(s) = \int_0^\infty f(it)\,t^{s-1}\,dt = \int_0^1 f(it)\,t^{s-1}\,dt + \int_1^\infty f(it)\,t^{s-1}\,dt.$$

Now using the substitution $t = 1/u$ on the first integral we have

$$\int_0^1 f(it)\,t^{s-1}\,dt = -\int_\infty^1 f\left(-\frac{1}{iu}\right)\,u^{-s-1}\,du = i^k \int_1^\infty f(iu)\,u^{k-s-1}\,du.$$

So we have that

$$(2\pi)^{-s} L(f,s)\,\Gamma(s) = i^k \int_1^\infty f(it)\,t^{k-s-1}\,dt + \int_1^\infty f(it)\,t^{s-1}\,dt.$$

Note that this integral is entire since $f(it)$ has exponential decay as $t \to \infty$ because f is a cusp form. Thus, we can extend $L(f,s)$ to an entire function. Returning to our line of thought, by replacing s with $k - s$ and multiplying through by i^k we have

$$i^k\,(2\pi)^{-(k-s)}\,L(f,k-s)\,\Gamma(k-s) = i^{2k} \int_1^\infty f(it)\,t^{s-1}\,dt + i^k \int_1^\infty f(it)\,t^{k-s-1}\,dt.$$

We know that k must be even since f is a cusp form, thus $i^{2k} = 1$. So we have that

$$\Lambda(f,s) = (2\pi)^{-s}\,\Gamma(s)\,L(f,s) = i^k\,(2\pi)^{-(k-s)}\,\Gamma(k-s)\,L(f,k-s) = i^k\,\Lambda(f,k-s)$$

as desired. $\square$

2.4 Hecke Operators and the Petersson Inner Product

Lemma 2.30 (Double Coset Motivation Hecke Operators). *We have that*

$$\{\gamma \in M_2(\mathbb{Z}) : \det \gamma = n,\ N \mid \gamma_c\} = \bigsqcup_{a^2 b = n} \Gamma_0(N) \begin{pmatrix} ab & 0 \\ 0 & a \end{pmatrix} \Gamma_0(N) = \bigsqcup_{\delta \in \Delta_n^N} \Gamma_0(N)\, \delta$$

where $\Delta_n^N = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : ad = n,\, b \bmod d,\, (a, N) = 1\}$. *This follows naturally by considering the Smith normal form and Hermite form of matrices.*

Definition 2.31 (Hecke Operators). *We define the Hecke operators acting on function* $f \in M_k(\Gamma_0(N))$ *as*

$$(T_n f)(z) = n^{k/2-1} \sum_{\delta \in \Delta_n^N} f|_k \delta(z).$$

Remark. *One can think of Hecke operators as an averaging operator.*

Lemma 2.32 (Hecke Operators as Indexed Sum). *The Hecke operators act on functions* $f \in M_k(\Gamma_0(N))$ *as*

$$(T_n f)(z) = n^{k-1} \sum_{ad=n} \sum_{b \bmod d} \chi_0^N(a)\, d^{-k}\, f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} z\right).$$

Proof. Trivial. ☐

Lemma 2.33 (Hecke Operators on Fourier Series). *If* $f \in M_k(\Gamma_0(N))$ *has Fourier expansion*

$$f(z) = \sum_{0 \le m} c_m e^{2\pi i m z} \qquad \text{then} \qquad (T_n f)(z) = \sum_{0 \le m} \sum_{d|(m,n)} \chi_0^N(d)\, d^{k-1}\, c_{mn/d^2}\, e^{2\pi i m z}.$$

Proof. By the previous lemma we have that

$$\begin{aligned}(T_n f)(z) &= n^{k-1} \sum_{ad=n} \sum_{b \bmod d} \chi_0^N(a)\, d^{-k} \left(\sum_{0 \le m} c_m e^{2\pi i m a z/d} e^{2\pi i m b/d}\right) \\ &= n^{k-1} \sum_{0 \le m} \sum_{ad=n} \chi_0^N(a)\, c_m \frac{e^{2\pi i m a z/d}}{d^{k-1}} \left(\frac{1}{d} \sum_{b \bmod d} e^{2\pi i m b/d}\right).\end{aligned}$$

Now note that

$$\frac{1}{d} \sum_{b \bmod d} e^{2\pi i m b/d} = 1 \qquad \text{if } d \mid m, \text{ and is 0 otherwise.}$$

Thus, noting that $a = n/d$ we now have

$$\begin{aligned}(T_n f)(z) &= n^{k-1} \sum_{0 \le m} \sum_{ad=n} \chi_0^N(a)\, c_m \frac{e^{2\pi i m a z/d}}{d^{k-1}} \left(\frac{1}{d} \sum_{b \bmod d} e^{2\pi i m b/d}\right) \\ &= n^{k-1} \sum_{0 \le m} \sum_{\substack{ad=n \\ d|m}} \chi_0^N(a)\, c_m \frac{e^{2\pi i m a z/d}}{d^{k-1}} = n^{k-1} \sum_{0 \le m} \sum_{d|(m,n)} \chi_0^N(n/d)\, c_m \frac{e^{2\pi i m n z/d^2}}{d^{k-1}}.\end{aligned}$$

Since $d \mid (m, n)$ we know that $d \mid m$ and thus there exists b such that $bd = m$ for all d. Using this we have

$$(T_n f)(z) = n^{k-1} \sum_{0 \le m} \sum_{d|(m,n)} \chi_0^N(n/d)\, c_m \frac{e^{2\pi i m n z/d^2}}{d^{k-1}} = n^{k-1} \sum_{0 \le b} \sum_{d|n} \chi_0^N(n/d)\, c_{bd} \frac{e^{2\pi i b n z/d}}{d^{k-1}}.$$

Likewise for $d \mid n$ we know there exists a such that $ad = n$ for all d. Using this we have

$$(T_n f)(z) = n^{k-1} \sum_{0 \le b} \sum_{d|n} \chi_0^N(n/d)\, c_{bd} \frac{e^{2\pi i b n z/d}}{d^{k-1}} = \sum_{0 \le b} \sum_{a|n} \chi_0^N(a)\, a^{k-1}\, c_{bn/a}\, e^{2\pi i a b z}.$$

Now re-indexing with $ab = m$ we have

$$(T_n f)(z) = \sum_{0\le b}\sum_{a|n} \chi_0^N(a)\, a^{k-1}\, c_{bn/a}\, e^{2\pi i abz} = \sum_{0\le m}\sum_{a|(m,n)} \chi_0^N(a)\, a^{k-1}\, c_{mn/a^2}\, e^{2\pi i mz}$$

exactly as desired. ☐

Lemma 2.34 (Hecke Operators are Multiplicative)**.** *For all $(m,n) = 1$, we have that $T_{mn} = T_m T_n$.*

Proof. By the above lemma we know that

$$(T_{mn} f)(z) = \sum_{0\le a}\sum_{d|(a,mn)} \chi_0^N(d)\, d^{k-1} c_{amn/d^2} e^{2\pi i az}.$$

Similarly, we have that

$$(T_n f)(z) = \sum_{0\le b}\sum_{d'|(b,n)} \chi_0^N(d')\, d'^{k-1} c_{bn/d^2} e^{2\pi i bz} = \sum_{0\le b} c'_b e^{2\pi i bz} \qquad \text{where} \qquad c'_b = \sum_{d'|(b,n)} \chi_0^N(d')\, d'^{k-1} c_{bn/d'^2}.$$

Thus, applying T_m to this new Fourier expansion gives

$$(T_m T_n f)(z) = \sum_{0\le a}\sum_{d|(a,m)} \chi_0^N(d)\, d^{k-1} c'_{am/d^2} e^{2\pi i az} = \sum_{0\le a}\sum_{d|(a,m)}\sum_{d'|(am/d^2,n)} \chi_0^N(dd')\, (dd')^{k-1} c_{amn/(dd')^2} e^{2\pi i az}$$

Now note that $d' \mid am/d^2$, thus $d' \mid am$. However, since $(m,n) = 1$ and $d' \mid n$, we know that $d' \nmid m$; thus, $d \mid a$. So this can be re-indexed with $d' \mid (a,n)$ instead.

$$\begin{aligned}(T_m T_n f)(z) &= \sum_{0\le a}\sum_{d|(a,m)}\sum_{d'|(am/d^2,n)} \chi_0^N(dd')\, (dd')^{k-1} c_{amn/(dd')^2} e^{2\pi i az} \\ &= \sum_{0\le a}\sum_{d|(a,m)}\sum_{d'|(a,n)} \chi_0^N(dd')\, (dd')^{k-1} c_{amn/(dd')^2} e^{2\pi i az}\end{aligned}$$

Now since $d \mid a$ and $d' \mid n$, we have that $dd' \mid an$. Likewise, since $d \mid m$ and $d' \mid a$, we have that $dd' \mid am$. Now since $(m,n) = 1$, we know there exists x and y such that $xm + yn = 1$. So, we have that $dd' \mid x(am) + y(an)$ with $x(am) + y(an) = a(xm + yn) = a$; thus, $dd' \mid a$. Trivially we have that $dd' \mid mn$ also. So this can be re-indexed by sending $dd' \mapsto d$ where $d \mid (a, mn)$ instead.

$$\begin{aligned}(T_m T_n f)(z) &= \sum_{0\le a}\sum_{d|(a,m)}\sum_{d'|(a,n)} \chi_0^N(dd')\, (dd')^{k-1} c_{amn/(dd')^2} e^{2\pi i az} \\ &= \sum_{0\le a}\sum_{d|(a,mn)} \chi_0^N(d)\, d^{k-1} c_{amn/d^2} e^{2\pi i az} = (T_{mn} f)(z)\end{aligned}$$

as desired. ☐

Corollary 2.35 (Normalized Eigenforms of Hecke Operators Have Multiplicative Fourier Coefficients)**.** *If $f \in S_k(\Gamma_0(N))$ has Fourier expansion*

$$f(z) = \sum_{1\le m} c_m e^{2\pi i mz}$$

with $c_1 = 1$ and there exists λ_n such that $T_n f = \lambda_n f$ for all n; then, $c_{mn} = c_m c_n$ for all $(m,n) = 1$.

Proof. Suppose that $f \in M_k$ has λ_n such that $T_n f = \lambda_n f$ for all n. Noting that

$$(T_n f)(z) = \sum_{0\le m}\sum_{d|(m,n)} \chi_0^N(d)\, d^{k-1} c_{mn/d^2} e^{2\pi i mz}$$

and matching the $m = 1$ Fourier coefficients, we have $c_n = \lambda_n$ for all n. Thus, for $(m,n) = 1$ we have

$$c_{mn} f = \lambda_{mn} f = T_{mn} f = T_m T_n f = \lambda_m \lambda_n f = c_m c_n f$$

dividing through by f completes the proof. ☐

Lemma 2.36 (Hecke Operator Composition Identity)**.** *We have that*

$$T_m T_n = \sum_{d|(m,n)} \chi_0^N(d)\, d^{k-1}\, T_{mn/d^2}$$

for all m and n.

Proof. Suppose that this identity holds for m and n prime powers with the same base. Now let p and q be distinct primes and note

$$\begin{aligned}&(p^a, q^b) = (q^b, p^c) = (p^c, q^d) = 1\\ &\qquad \implies T_{p^a q^b} T_{p^c q^d} = T_{p^a} T_{q^b} T_{p^c} T_{q^d} = T_{p^a} T_{q^b p^c} T_{q^d} = T_{p^a} T_{p^c q^b} T_{q^d} = T_{p^a} T_{p^c} T_{q^b} T_{q^d}.\end{aligned}$$

Thus,

$$\begin{aligned}T_{p^a q^b} T_{p^c q^d} = (T_{p^a} T_{p^c})(T_{q^b} T_{q^d}) &= \left(\sum_{d|(p^a,p^c)} \chi_0^N(d)\, d^{k-1} T_{p^a p^c/d^2}\right)\left(\sum_{d'|(q^b,q^d)} \chi_0^N(d')\, d'^{k-1} T_{q^b q^d/d'^2}\right)\\ &= \sum_{d|(p^a,p^c)} \sum_{d'|(q^b,q^d)} \chi_0^N(dd')\, (dd')^{k-1} T_{p^a p^c/d^2} T_{q^b q^d/d'^2}.\end{aligned}$$

But now note that $(p^a p^c/d^2, q^b q^d/d'^2) = 1$, thus

$$T_{p^a q^b} T_{p^c q^d} = \sum_{d|(p^a,p^c)} \sum_{d'|(q^b,q^d)} \chi_0^N(dd')\, (dd')^{k-1} T_{p^a p^c/d^2} T_{q^b q^d/d'^2} = \sum_{d|(p^a,p^c)} \sum_{d'|(q^b,q^d)} \chi_0^N(dd')\, (dd')^{k-1} T_{p^a q^b p^c q^d/(dd')^2}.$$

Now we can re-index by sending $dd' \mapsto d$ with $d \mid (p^a q^b, p^c q^d)$.

$$T_{p^a q^b} T_{p^c q^d} = \sum_{d|(p^a,p^c)} \sum_{d'|(q^b,q^d)} \chi_0^N(dd')\, (dd')^{k-1} T_{p^a q^b p^c q^d/(dd')^2} = \sum_{d|(p^a q^b, p^c q^d)} \chi_0^N(d)\, d^{k-1} T_{p^a q^b p^c q^d/d^2}.$$

Inductively, and applying the fundamental theorem of arithmetic, we have that this identity then holds for all pairs of integers.

Thus,we need to prove this identity holds for prime powers with the same base.

We first prove the base case, $T_p T_{p^n} = T_{p^{n+1}} + \chi_0^N(p)\, p^{k-1} T_{p^{n-1}}$. Note that

$$(T_{p^n} f)(z) = \sum_{0\le b} \sum_{d'|(b,p^n)} \chi_0^N(d)\, d'^{k-1} c_{bp^n/d'^2} e^{2\pi i b z} = \sum_{0 \le b} c'_b e^{2\pi i b z} \qquad \text{where} \qquad c'_b = \sum_{d|(b,p^n)} \chi_0^N(d)\, d^{k-1} c_{bp^n/d'^2}.$$

Thus, applying T_p to this new Fourier expansion gives

$$(T_p T_{p^n} f)(z) = \sum_{0\le a} \sum_{d|(a,p)} \chi_0^N(d)\, d^{k-1} c'_{ap/d^2}\, e^{2\pi i a z} = \sum_{0\le a} \sum_{d|(a,p)} \sum_{d'|(ap/d^2,p^n)} \chi_0^N(dd')\, (dd')^{k-1} c_{ap^{n+1}/(dd')^2}\, e^{2\pi i a z}.$$

Now note that we get a $d = 1$ term from every a, and a $d = p$ term from every a which is a multiple of p; thus,

$$\begin{aligned}(T_p T_{p^n} f)(z) &= \sum_{0\le a} \sum_{d'|(ap,p^n)} \chi_0^N(d')\, d'^{k-1} c_{ap^{n+1}/d'^2} e^{2\pi i a z}\\ &\quad + \sum_{0\le a} \sum_{d'|(a,p^n)} \chi_0^N(pd')\, (pd')^{k-1} c_{ap^n/d'^2} e^{2\pi i a p z}.\end{aligned}$$

Focusing on the first double sum and sending $d' \mapsto pd$ we have

$$\begin{aligned}&\sum_{0\le a} \sum_{d'|(ap,p^n)} \chi_0^N(d')\, d'^{k-1} c_{ap^{n+1}/d'^2} e^{2\pi i a z}\\ &\quad = \chi_0^N(p)\, p^{k-1} \sum_{0\le a} \sum_{d|(a,p^{n-1})} \chi_0^N(d)\, d^{k-1} c_{ap^{n-1}/d^2} e^{2\pi i a z} = \chi_0^N(p)\, p^{k-1} (T_{p^{n-1}} f)(z).\end{aligned}$$

Focusing on the second double sum and sending $pa \mapsto a$ and $pd' \mapsto d$ we have

$$\begin{aligned}\sum_{0\le a}\sum_{d|(a,p^n)} &\chi_0^N(pd')\,(pd')^{k-1}c_{ap^n/d'^2}e^{2\pi iapz}\\ &=\sum_{0\le a}\sum_{d|(a,p^{n+1})}\chi_0^N(d)\,d^{k-1}c_{ap^{n+1}/d^2}e^{2\pi iaz}=(T_{p^{n+1}}f)(z)\end{aligned}$$

Thus $T_pT_{p^n}=T_{p^{n+1}}+\chi_0^N(p)\,p^{k-1}T_{p^{n-1}}$ as desired.

Now we will proceed via strong induction. We need to show that

$$T_{p^{m+1}}T_{p^n}=\sum_{d|(p^{m+1},p^n)}\chi_0^N(d)\,d^{k-1}T_{p^{m+1}p^n/d^2}$$

assuming it holds for all lower cases. WLOG, suppose that $m\le n$. Now, note that

$$\begin{aligned}T_pT_{p^m}T_{p^n}=T_p\sum_{d|(p^m,p^n)}\chi_0^N(d)\,d^{k-1}T_{p^mp^n/d^2}&=\sum_{d|(p^m,p^n)}\chi_0^N(d)\,d^{k-1}T_pT_{p^mp^n/d^2}\\ &=\sum_{d|(p^m,p^n)}\chi_0^N(d)\,d^{k-1}(T_{p^{m+1}p^n/d^2}+\chi_0^N(p)\,p^{k-1}T_{p^{m-1}p^n/d^2})\\ &=\sum_{d|(p^m,p^n)}\chi_0^N(d)\,d^{k-1}T_{p^{m+1}p^n/d^2}+\sum_{d|(p^m,p^n)}\chi_0^N(pd)\,(pd)^{k-1}T_{p^{m-1}p^n/d^2}.\end{aligned}$$

Extracting the $d=1$ term from the first sum we have

$$\begin{aligned}T_pT_{p^m}T_{p^n}&=\sum_{d|(p^m,p^n)}\chi_0^N(d)\,d^{k-1}T_{p^{m+1}p^n/d^2}+\sum_{d|(p^m,p^n)}\chi_0^N(pd)\,(pd)^{k-1}T_{p^{m-1}p^n/d^2}\\ &=T_{p^{m+1}p^n}+\chi_0^N(p)\,p^{k-1}\sum_{d|(p^{m-1},p^n)}\chi_0^N(d)\,d^{k-1}T_{p^{m-1}p^n/d^2}+\sum_{d|(p^m,p^n)}\chi_0^N(pd)\,(pd)^{k-1}T_{p^{m-1}p^n/d^2}\\ &=T_{p^{m+1}p^n}+\chi_0^N(p)\,p^{k-1}T_{p^{m-1}}T_{p^n}+\sum_{d|(p^m,p^n)}\chi_0^N(pd)\,(pd)^{k-1}T_{p^{m-1}p^n/d^2}.\end{aligned}$$

Thus,

$$\begin{aligned}(T_pT_{p^m}-\chi_0^N(p)\,p^{k-1}T_{p^{m-1}})\,T_{p^n}&=T_{p^{m+1}p^n}+\sum_{d|(p^m,p^n)}\chi_0^N(pd)\,(pd)^{k-1}T_{p^{m-1}p^n/d^2}\\ \implies T_{p^{m+1}}T_{p^n}&=\sum_{d|(p^{m+1},p^n)}\chi_0^N(d)\,d^{k-1}T_{p^{m+1}p^n/d^2}.\end{aligned}$$

Thus the proof is complete.

□

Corollary 2.37 (Hecke Operators are Commutative)**.** *For all m and n we have that $T_mT_n=T_nT_m$.*

Proof. Since the RHS in the previous identity is the same if m and n swap, commutativity immediately follows. □

Lemma 2.38 (Hecke Operators Preserve $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$)**.** *We have that $T_n: M_k(\Gamma_0(N))\to M_k(\Gamma_0(N))$ and $T_n: S_k(\Gamma_0(N))\to S_k(\Gamma_0(N))$.*

Proof. For $\gamma\in\Gamma_0(N)$ note that

$$(T_nf)|_k\gamma=n^{k/2-1}\sum_{\delta\in\Delta_n^N}f|_k\delta|_k\gamma=n^{k/2-1}\sum_{\delta\in\Delta_n^N}f|_k\delta\gamma.$$

Now note that $\Delta_n^N=\Gamma_0(N)\,\Delta_n^N=\Delta_n^N\,\Gamma_0(N)$. So there exists $\gamma'\in\Gamma_0(N)$ and $\delta'\in\Delta_n^N$ such that $\delta\gamma=\gamma'\delta'$ and indeed this map $\delta\mapsto\delta'$ forms a bijection across elements of Δ_n^N. Thus,

$$(T_nf)|_\gamma=n^{k/2-1}\sum_{\delta\in\Delta_n^N}f|_k\delta\gamma=n^{k/2-1}\sum_{\delta'\in\Delta_n^N}f|_k\gamma'\delta'=n^{k/2-1}\sum_{\delta'\in\Delta_n^N}f|_k\gamma'|_k\delta'.$$

So by the modularity of f we have that

$$(T_n f)|_k \gamma = n^{k/2-1} \sum_{\delta' \in \Delta_n^N} f|_k \gamma'|_k \delta' = n^{k/2-1} \sum_{\delta' \in \Delta_n^N} f|_k \delta' = T_n f.$$

Thus $T_n f$ is modular and so $T_n : M_k(\Gamma_0(N)) \to M_k(\Gamma_0(N))$.

Now if $f \in S_k(\Gamma_0(N))$ then if we let

$$(T_n f)(z) = \sum_{0 \le m} \sum_{d|(m,n)} \chi_0^N(d)\, d^{k-1} c_{mn/d^2} e^{2\pi i m z} = \sum_{0 \le m} c'_m e^{2\pi i m z}$$

then we have that

$$c'_0 = \sum_{d|(0,n)} \chi_0^N(d)\, d^{k-1} c_0.$$

But since $f \in S_k$ we know that $c_0 = 0$; thus, $c'_0 = 0$ and we have that $f \in S_k(\Gamma_0(N))$ also. □

Corollary 2.39 (τ is Multiplicative)**.** *For all $(m,n) = 1$ we have that $\tau(mn) = \tau(m)\,\tau(n)$.*

Proof. Since $S_{12} = \Delta \cdot \mathbb{C}$ and $T_n : S_{12} \to S_{12}$, we know there exists λ_n such that $T_n \Delta = \lambda_n \Delta$ for all n. Thus Δ is an eigenform of the Hecke operators and we have that $\tau(mn) = \tau(m)\,\tau(n)$ for all $(m,n) = 1$. □

Definition 2.40 (Petersson Inner Product)**.** *For modular forms $f, g \in M_k(\Gamma_0(N))$ we define the Petersson inner product*

$$\langle f, g \rangle = \int_{\Gamma_0(N)/\mathcal{H}} f(z)\, \overline{g(z)}\, y^{k-2}\, dx\, dy$$

where $\Gamma_0(N)/\mathcal{H}$ is the fundamental domain of $\mathcal{H}$ under the action of $\Gamma_0(N)$.

Lemma 2.41 (Poincare Series and the Petersson Inner Product)**.** *If $f \in M_k(\Gamma_0(N))$ we have that*

$$\langle f, P_{m,k} \rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\, \hat{f}(m)$$

where $\hat{f}$ is the Fourier transform of f.

Proof. Note that

$$\begin{aligned} \langle f, P_{m,k} \rangle &= \int_{\Gamma_0(N)/\mathcal{H}} y^k f(z) \sum_{\gamma \in \Gamma_\infty/\Gamma_0(N)} \overline{e(mz)|_k \gamma}\, \frac{dx\, dy}{y^2} \\ &= \sum_{\gamma \in \Gamma_\infty/\Gamma_0(N)} \int_{\Gamma_0(N)/\mathcal{H}} y^k f(z) \overline{j(\gamma, z)^{-k}\, e(m\gamma z)}\, \frac{dx\, dy}{y^2}. \end{aligned}$$

Now since our measure is $\Gamma_0(N)$ invariant we make a substitution on the integral

$$\langle f, P_{m,k} \rangle = \sum_{\gamma \in \Gamma_\infty/\Gamma_0(N)} \int_{\gamma(\Gamma_0(N)/\mathcal{H})} \mathrm{Im}(\gamma^{-1} z)^k f(\gamma^{-1} z)\, \overline{j(\gamma, \gamma^{-1} z)^{-k}\, e(mz)}\, \frac{dx\, dy}{y^2}.$$

Now doing some quick calculations we have that

$$\mathrm{Im}(\gamma^{-1} z)^k = \frac{y^k}{|j(\gamma^{-1}, z)|^{2k}} \qquad \text{and} \qquad j(\gamma, \gamma^{-1} z)^{-k} = j(\gamma^{-1}, z)^k;$$

thus, using the modularity of f we have that

$$\begin{aligned} \langle f, P_{m,k} \rangle &= \sum_{\gamma \in \Gamma_\infty/\Gamma_0(N)} \int_{\gamma(\Gamma_0(N)/\mathcal{H})} \frac{y^k}{|j(\gamma^{-1}, z)|^{2k}}\, j(\gamma^{-1}, z)^k f(z)\, \overline{j(\gamma^{-1}, z)^k\, e(mz)}\, \frac{dx\, dy}{y^2} \\ &= \sum_{\gamma \in \Gamma_\infty/\Gamma_0(N)} \int_{\gamma(\Gamma_0(N)/\mathcal{H})} y^k f(z) \overline{e(mz)}\, \frac{dx\, dy}{y^2} = \int_{\Gamma_\infty/\mathcal{H}} y^k f(z)\, e(-mx)\, e(imy)\, \frac{dx\, dy}{y^2}. \end{aligned}$$

But now note that the action of Γ_∞ on $\mathcal{H}$ is translation by some integer amount in the real component. Thus,

$$\langle f, P_{m,k}\rangle = \int_0^\infty y^k\, e(imy) \int_0^1 f(z)\, e(-mx)\, \frac{dx\, dy}{y^2}.$$

Now we recall the Fourier expansion of f,

$$\begin{aligned}\langle f, P_{m,k}\rangle &= \sum_{0\le n} a_n \int_0^\infty y^k e(imy+iny) \int_0^1 e(nx-mx)\, \frac{dx\, dy}{y^2}\\ &= \sum_{0\le n} a_n\delta_{mn} \int_0^\infty y^k\, e(imy+iny)\, \frac{dy}{y^2} = a_m \int_0^\infty y^{k-2}\, e^{-4\pi my}\, dy = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\, \hat{f}(m),\end{aligned}$$

exactly as desired. □

Theorem 2.42 (Poincare Series Span the Space $S_k(\Gamma_0(N))$)**.** *We have that $S_k(\Gamma_0(N)) = \mathrm{Span}\{P_{m,k} : 1 \le m\}$.*

Proof. Let $S_k'(\Gamma_0(N)) = \mathrm{Span}\{P_{m,k} : 1 \le m\}$. Now if we suppose that f is in the orthogonal complement of $S_k'(\Gamma_0(N))$, then $\langle f, P_{m,k}\rangle = 0$ for all $1 \le m$. Thus, $\hat{f}(m) = 0$ for all $1 \le m$ by the above. Thus $f \equiv 0$, and $S_k(\Gamma_0(N)) = S_k'(\Gamma_0(N))$ as desired. □

Lemma 2.43 (Hecke Operator and Poincare Series Symmetry Lemma I)**.** *We have that*

$$\chi_0^N(m)\, m^{k-1} T_n P_{m,k} = \chi_0^N(n)\, n^{k-1} T_m P_{n,k}.$$

Proof. Note that

$$\begin{aligned}\chi_0^N(m)\, m^{k-1}(T_nP_{m,k})(z) &= \chi_0^N(m)\,(mn)^{k-1} \sum_{ad=n}\sum_{b \bmod d}\sum_{\gamma\in\Gamma_\infty/\Gamma_0(N)} \chi_0^N(a)\, d^{-k} e\left(m\begin{pmatrix} a & b\\ 0 & d\end{pmatrix} z\right)\Bigg|_k \gamma\\ &= \chi_0^N(m)\,(mn)^{k-1} \sum_{ad=n}\sum_{b \bmod d}\sum_{\gamma\in\Gamma_\infty/\Gamma_0(N)} \chi_0^N(a)\, d^{-k}\, j(\gamma,z)^{-k}\, e\left(m\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\gamma z\right)\\ &= (mn)^{k-1}\sum_{ad=n}\sum_{\gamma\in\Gamma_\infty/\Gamma_0(N)} \frac{\chi_0^N(ma)\, e(ma\gamma z/d)}{d^{k-1}\, j(\gamma,z)^k}\left(\frac{1}{d}\sum_{b \bmod d} e(mb/d)\right).\end{aligned}$$

Now note that

$$\frac{1}{d}\sum_{b \bmod d} e(mb/d) \qquad \text{if } d \mid m, \text{ and is 0 otherwise.}$$

Thus, noting that $a = n/d$ we now have

$$\begin{aligned}\chi_0^N(m)\, m^{k-1}(T_nP_{m,k})(z) =&\ (mn)^{k-1}\sum_{ad=n}\sum_{\gamma\in\Gamma_\infty/\Gamma_0(N)} \frac{\chi_0^N(ma)\, e(ma\gamma z/d)}{d^{k-1}\, j(\gamma,z)^k}\left(\frac{1}{d}\sum_{b \bmod d} e(mb/d)\right)\\ &= (mn)^{k-1}\sum_{\substack{ad=n\\ d\mid m}}\sum_{\gamma\in\Gamma_\infty/\Gamma_0(N)} \frac{\chi_0^N(ma)\, e(ma\gamma z/d)}{d^{k-1}\, j(\gamma,z)^k}\\ &= (mn)^{k-1}\sum_{d\mid(m,n)}\sum_{\gamma\in\Gamma_\infty/\Gamma_0(N)} \frac{\chi_0^N(mn/d)\, e(mn\gamma z/d^2)}{d^{k-1}\, j(\gamma,z)^k}.\end{aligned}$$

This expression is symmetric under $(m,n) \mapsto (n,m)$; so we have that

$$\chi_0^N(m)\, m^{k-1}\, T_nP_{m,k} = \chi_0^N(n)\, n^{k-1}\, T_mP_{n,k}$$

as desired. □

Lemma 2.44 (Hecke Operator and Poincare Series Symmetry Lemma II)**.** *For $f \in M_k(\Gamma_0(N))$ we have that*

$$m^{k-1}\langle T_nf,\, P_{m,k}\rangle = n^{k-1}\langle T_mf,\, P_{n,k}\rangle.$$

Proof. Note that

$$T_m f = \sum_{0\leq a}\sum_{d|(a,m)} \chi_0^N(d)\, d^{k-1} c_{am/d^2}\, e^{2\pi iaz} \qquad \text{and} \qquad T_n f = \sum_{0\leq a}\sum_{d|(a,n)} \chi_0^N(d)\, d^{k-1} c_{an/d^2}\, e^{2\pi iaz}.$$

Thus we immediately have that $\widehat{(T_m f)}(n) = \widehat{(T_n f)}(m)$. Thus from our previous lemma,

$$\langle T_m f, P_{n,k}\rangle \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} = \widehat{(T_m f)}(n) = \widehat{(T_n f)}(m) = \langle T_n f, P_{m,k}\rangle \frac{(4\pi m)^{k-1}}{\Gamma(k-1)}.$$

Removing the scalar factors from both sides completes our proof. □

Theorem 2.45 (Hecke Operators are Self-Adjoint w.r.t Petersson Inner Product on S_k)**.** *For all n and $f, g \in S_k$ we have that $\langle T_n f, g\rangle = \langle f, T_n g\rangle$.*

Proof. Since $f, g \in S_k$ we know that $N = 1$ and so

$$m^{k-1} T_n P_{m,k} = \chi_0^N(m)\, m^{k-1} T_n P_{m,k} = \chi_0^N(n)\, n^{k-1} T_m P_{n,k} = n^{k-1} T_m P_{n,k}.$$

Thus we have that

$$\langle T_n P_{r,k}, P_{s,k}\rangle = \left(\frac{n}{s}\right)^{k-1} \langle T_s P_{r,k}, P_{n,k}\rangle = \left(\frac{n}{r}\right)^{k-1} \langle T_r P_{s,k}, P_{n,k}\rangle = \langle T_n P_{s,k}, P_{r,k}\rangle = \overline{\langle P_{r,k}, T_n P_{s,k}\rangle}.$$

But now since the Fourier coefficients of $P_{r,k}$ and $P_{s,k}$ are real, we know that

$$\langle T_n P_{r,k}, P_{s,k}\rangle = \overline{\langle P_{r,k}, T_n P_{s,k}\rangle} = \langle P_{r,k}, T_n P_{s,k}\rangle.$$

Thus, for all n, T_n is self adjoint w.r.t the Petersson inner product. □

Lemma 2.46 (Spectral Theorem)**.** *If $T, T' : V \to V$ are commuting complex valued linear operations which satisfy $\langle Tv, v'\rangle = \langle v, Tv'\rangle$ and $\langle T'v, v'\rangle = \langle v, T'v'\rangle$ for all $v, v' \in V$; then there exists a simultaneous eigenbasis of T and T'.*

Proof. Since T has a characteristic polynomial, we know there exists at least one eigenvector $v \neq 0$ such that $Tv = \lambda v$ with $\lambda \neq 0$. Now if $v \neq 0$ is an arbitrary eigenvector with associated eigenvalue $\lambda \neq 0$, then

$$\lambda\langle v, v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle = \overline{\lambda}\langle v, v\rangle \implies 0 = (\lambda - \overline{\lambda})\langle v, v\rangle.$$

Since $v \neq 0$, we know that $\langle v, v\rangle \neq 0$; thus, $\lambda \in \mathbb{R}$. Now if $v, v' \neq 0$ are arbitrary eigenvectors with associated eigenvalues $0 \neq \lambda \neq \lambda' \neq 0$, then

$$\lambda\langle v, v'\rangle = \langle Tv, v'\rangle = \langle v, Tv'\rangle = \overline{\lambda'}\langle v, v'\rangle \implies 0 = (\lambda - \overline{\lambda'})\langle v, v'\rangle.$$

Since $\lambda' \in \mathbb{R}$ and $\lambda \neq \lambda'$ we know that $\langle v, v'\rangle = 0$. Thus, we have that

$$V = \bigoplus_{\lambda\in\mathbb{R}} \mathbb{C}E_\lambda$$

where $E_\lambda = \{v \in V : Tv = \lambda v\}$. Now note that

$$T(T'E_\lambda) = T'TE_\lambda = T'\lambda E_\lambda = \lambda(T'E_\lambda) \implies T'E_\lambda = \lambda' E_\lambda$$

for some $\lambda' \in \mathbb{C}$. Thus you can find a simultaineous eigenbasis of T and T'. □

Corollary 2.47 (There Exists a Basis of S_k which are Eigenforms of Hecke Operators)**.** *There exists a basis of S_k such that every element is an Eigenform of all Hecke operators T_n.*

Proof. Applying the spectral theorem immediately gives the desired result. □

Theorem 2.48 (Slash Operator is an Isometry of $S_k(\Gamma_0(N))$ w.r.t Petersson Inner Product)**.** *We have that $\langle f|_k\gamma, g|_k\gamma\rangle = \langle f, g\rangle$ for all $\gamma \in \mathrm{GL}_2(\mathbb{R})$ and $f, g \in S_k(\Gamma_0(N))$.*

Proof. For every $\gamma \in \mathrm{GL}_2(\mathbb{R})$ there exists $\gamma' \in \mathrm{SL}_2(\mathbb{R})$ such that $f|_k\gamma = f|_k\gamma'$. Thus it is sufficient to prove the identity for $\mathrm{SL}_2(\mathbb{R})$. Note that

$$\langle f|_k\gamma,\, g|_k\gamma\rangle = \int_{\Gamma_0(N)/\mathcal{H}} y^k\, f|_k\gamma(z)\,\overline{g|_k\gamma(z)}\,\frac{dx\,dy}{y^2} = \int_{\Gamma_0(N)/\mathcal{H}} \frac{y^k}{j(\gamma,z)^{2k}}\, f(\gamma z)\,\overline{g(\gamma z)}\,\frac{dx\,dy}{y^2}.$$

Now since our measure is $\mathrm{GL}_2^+(\mathbb{R})$ invariant we make a substitution on the integral

$$\langle f|_k\gamma,\, g|_k\gamma\rangle = \int_{\gamma(\Gamma_0(N)/\mathcal{H})} \frac{\mathfrak{Im}(\gamma^{-1}z)^k}{j(\gamma,\gamma^{-1}z)^{2k}}\, f(z)\,\overline{g(z)}\,\frac{dx\,dy}{y^2}.$$

Now doing some quick calculations we have that

$$\mathfrak{Im}(\gamma^{-1}z)^k = \frac{y^k}{j(\gamma^{-1},z)^{2k}} \qquad \text{and} \qquad j(\gamma,\gamma^{-1}z)^{-2k} = j(\gamma^{-1},z)^{2k};$$

thus we have that

$$\langle f|_k\gamma,\, g|_k\gamma\rangle = \int_{\gamma(\Gamma_0(N)/\mathcal{H})} y^k\, f(z)\,\overline{g(z)}\,\frac{dx\,dy}{y^2} = \langle f, g\rangle$$

since our integral is independent of our choice of fundamental domain. □

Theorem 2.49 (Hecke Operators are Self-Adjoint w.r.t Petersson Inner Product on $S_k(\Gamma_0(N))$)**.** *For all n and $f,\, g \in S_k(\Gamma_0(N))$ we have that $\langle T_n f, g\rangle = \chi_0^N(n)\,\langle f, T_n g\rangle$.*

Proof. For $ad = n$ and $b \bmod d$ let $\delta = \left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right)$ and $\delta' = \left(\begin{smallmatrix} d & -b \\ 0 & a\end{smallmatrix}\right)$. Now since the slash operator is an isometry we have

$$\langle f|_k\delta, g\rangle = \langle f|_k\delta|_k\delta', g|_k\delta'\rangle = \langle f|_k\left(\begin{smallmatrix} n & 0 \\ 0 & n\end{smallmatrix}\right), g|_k\delta'\rangle = \langle f, g|_k\delta'\rangle.$$

Now note that

$$T_n f = n^{k/2-1} \sum_{\delta\in\Delta_n^N} f|_k\delta = n^{k/2-1} \sum_{\delta\in\Delta_n^1} \chi_0^N(a)\, f|_k\delta.$$

Thus it follows that

$$\begin{aligned}\langle T_n f, g\rangle = n^{k/2-1} \sum_{\delta\in\Delta_n^1} \langle \chi_0^N(a)\, f|_k\delta, g\rangle &= n^{k/2-1} \sum_{\delta\in\Delta_n^1} \chi_0^N(a)\langle f|_k\delta, g\rangle \\ &= \chi_0^N(n)\, n^{k/2-1} \sum_{\delta\in\Delta_n^1} \overline{\chi_0^N(d)}\langle f, g|_k\delta'\rangle = \chi_0^N(n)\, n^{k/2-1} \sum_{\delta\in\Delta_n^1} \langle f, \chi_0^N(d)\, g|_k\delta'\rangle.\end{aligned}$$

Let $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right) \in \Gamma_0(N)$. Thus, by modularity $g = g|_k T$. So since $\delta \mapsto T\delta'$ is a bijection on Δ_n^1 we have that

$$\begin{aligned}\langle T_n f, g\rangle = \langle T_n f, g|_k T\rangle &= \chi_0^N(n)\, n^{k/2-1} \sum_{\delta\in\Delta_n^1} \langle f, \chi_0^N(d)\, g|_k T|_k\delta'\rangle \\ &= \chi_0^N(n)\, n^{k/2-1} \sum_{\delta\in\Delta_n^1} \langle f, \chi_0^N(d)\, g|_k T\delta'\rangle = \chi_0^N(n)\, n^{k/2-1} \sum_{\delta'\in\Delta_n^1} \langle f, \chi_0^N(d)\, g|_k\delta'\rangle = \chi_0^N(n)\,\langle f, T_n g\rangle.\end{aligned}$$

□

Corollary 2.50 (There Exists a Basis of $S_k(\Gamma_0(N))$ which are Eigenforms of Hecke Operators)**.** *There exists a basis of $S_k(\Gamma_0(N))$ such that every element is an Eigenform of all Hecke operators T_n with $(n, N) = 1$.*

Proof. Note that since

$$\langle T_n f, g\rangle = \chi_0^N(n)\,\langle f, T_n g\rangle$$

we know that T_n is self-adjoint w.r.t. the Petersson inner product for all $(n, N) = 1$. Applying the spectral theorem immediately gives the desired result. □

2.5 Atkin-Lehner Theory

Lemma 2.51 (Injection of Cusp Forms to Higher Level I). *If $f \in S_k(\Gamma_0(M))$ and $M \mid N$, then $f \in S_k(\Gamma_0(N))$.*

Proof. Note that $\Gamma_0(N)$ is a subgroup of $\Gamma_0(M)$ when $M \mid N$. Thus, $f \in S_k(\Gamma_0(M)) \implies f \in S_k(\Gamma_0(N))$. □

Lemma 2.52 (Injection of Cusp Forms to Higher Level II). *If $f(z) \in S_k(\Gamma_0(M))$ then $f(rz) \in S_k(\Gamma_0(rM))$.*

Proof. Let $g(z) = f(rz)$ and let $\gamma = \left(\begin{smallmatrix} a & b \\ crM & d \end{smallmatrix}\right) \in \Gamma_0(rM)$. Then we have that

$$g(\gamma z) = f(r\gamma z) = f\left(r \cdot \left(\frac{az+b}{crMz+d}\right)\right) = f\left(\frac{a(rz)+rb}{cM(rz)+d}\right) = f(\gamma'(rz))$$

where $\gamma' = \left(\begin{smallmatrix} a & rb \\ cM & d \end{smallmatrix}\right)$. Note that $\det \gamma' = \det \gamma = 1$ and that $cN \equiv 0 \bmod N$, thus $\gamma' \in \Gamma_0(N)$. So by the modularity of f on $\Gamma_0(N)$ we have that

$$g(\gamma z) = f(\gamma'(rz)) = j(\gamma', rz)^k f(rz) = (crMz+d)^k f(rz) = j(\gamma, z)^k g(z).$$

Thus g is modular on $\Gamma_0(rM)$ as desired. □

Corollary 2.53 (Injection of Cusp Forms to Higher Level III). *If $f(z) \in S_k(\Gamma_0(M))$, $M \mid N$, and $r \mid (N/M)$; then $f(rz) \in S_k(\Gamma_0(N))$.*

Proof. Simple consequence of the previous two lemmas. □

Definition 2.54 (Space of Oldforms). *We define the space of level N oldforms, denoted $S_k^{old}(\Gamma_0(N))$ as the cusp forms "coming from those of lower level." More precisely,*

$$S_k^{old}(\Gamma_0(N)) = Span\left\{ \bigcup_{\substack{M|N \\ M\neq N}} \bigcup_{r|(N/M)} \{f(rz) : f(z) \in S_k(\Gamma_0(M))\} \right\}.$$

Definition 2.55 (Space of Newforms). *The space of level N newforms, denoted $S_k^{new}(\Gamma_0(N))$, is the orthogonal complement of the space of oldforms with respect to the Petersson inner product.*

Lemma 2.56 (Hecke Operators Preserve $S_k^{\text{old}}(\Gamma_0(N))$ and $S_k^{\text{new}}(\Gamma_0(N))$). *We have that $T_n : S_k^{old}(\Gamma_0(N)) \to S_k^{old}(\Gamma_0(N))$ and $T_n : S_k^{new}(\Gamma_0(N)) \to S_k^{new}(\Gamma_0(N))$.*

Proof. Note that if $f \in S_k^{\text{old}}(\Gamma_0(N))$ then there exists an $M \mid N$ with $M \neq N$ such that $f \in S_k(\Gamma_0(M))$. Then we know $T_n f \in S_k(\Gamma_0(M))$ since Hecke operators preserve $S_k(\Gamma_0(M))$. But then we know that $T_n f \in S_k^{\text{old}}(\Gamma_0(N))$ by the definition of the space of oldforms.

Now if $f \in S_k^{\text{new}}(\Gamma_0(N))$ then we know that $T_n f \in S_k^{\text{new}}(\Gamma_0(N))$ since T_n preserves the space of oldforms and the space of newforms is the orthogonal complement of the space of oldforms. □

Definition 2.57 (Newforms). *A level N newform is an $f \in S_k^{new}(\Gamma_0(N))$ such that f is a normalized simultaneous eigenform of all Hecke operators T_n with $(n, N) = 1$.*

Theorem 2.58 (Multiplicity-One Theorem). *If f and g are newforms with the same Hecke eigenvalues λ_n for all $(n, N) = 1$, then $f = g$.*

Proof. If f is a newform then we know it is a normalized Hecke eigenform. Furthermore, if

$$f(z) = \sum_{1 \leq m} c_m e^{2\pi imz} \qquad \text{and} \qquad g(z) = \sum_{1 \leq m} c'_m e^{2\pi imz}$$

then note that, for $(n, N) = 1$ one has

$$\lambda_n f = T_n f = \sum_{0 \leq m} \sum_{d|(m,n)} \chi_0^N(d)\, d^{k-1} c_{mn/d^2} e^{2\pi imz}$$
$$\lambda_n g = T_n g = \sum_{0 \leq m} \sum_{d|(m,n)} \chi_0^N(d)\, d^{k-1} c_{mn/d^2} e^{2\pi imz}.$$

Matching the $m = 1$ coefficients we have that $\lambda_n = \lambda_n c_1 = c_n$ and $\lambda_n = \lambda_n c_1' = c_n'$ since $c_1 = 1$ and $c_1' = 1$ comes from f and g being normalized. Thus, $c_n = \lambda_n = c_n'$ for $(n, N) = 1$. Thus we have that

$$f - g = \sum_{(m,N)\neq 1} (c_m - c_m')\, e^{2\pi i m z} \in S_k^{\text{old}}(\Gamma_0(N)).$$

But note that we have expressed an oldform as a linear combination of newforms, and the space of oldforms is orthogonal to the space of newforms; so we know that $f - g = 0$. ☐

Corollary 2.59 (Newforms are Eigenforms of all Hecke Operators)**.** *A level N newform f is a normalized simultaneous eigenform of all Hecke operators T_n.*

Proof. Suppose $T_n f = \lambda_n f$ for all $(n, N) = 1$, and note that

$$T_n T_m f = T_m T_n f = T_m \lambda_n f = \lambda_n T_m f.$$

Thus, $T_m f$ is a Hecke eigenform of all $(n, N) = 1$. So we know there exists λ_m such that $\lambda_m g = T_m f$ where g is a newform. Then we have that

$$\lambda_m\, T_n g = T_n T_m f = \lambda_n T_m f = \lambda_m \lambda_n g \implies T_n g = \lambda_n g.$$

But since f and g are newforms with the same eigenvalues at $(n, N) = 1$ we know that $f = g$ by the multiplicity one theorem. Thus,

$$T_m f = \lambda_m g = \lambda_m f.$$

So f is a simultaneous eigenform of all Hecke operators T_n. ☐

Corollary 2.60 (There Exists a Basis of $S_k^{\text{new}}(\Gamma_0(N))$ which are Eigenforms of Hecke Operators)**.** *There exists a basis of $S_k^{new}(\Gamma_0(N))$ such that every element is an Eigenform of all Hecke operators T_n.*

Proof. Note that since

$$\langle T_n f, g\rangle = \chi_0^N(n)\langle f, T_n g\rangle$$

we know that T_n is self-adjoint w.r.t the Petersson inner product for all $(n, N) = 1$. Applying the spectral theorem and renomralizing tells us that we can construct a basis for $S_k^{\text{new}}(\Gamma_0(N))$ with all of the elements being newforms. By the above we know that newforms are Eigenforms of all Hecke operators. ☐

Lemma 2.61 (The Fricke Involution Preserve $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$)**.** *If we let $Wf = f|_k\omega$ where $\omega = \left(\begin{smallmatrix} 0 & -1/\sqrt{N} \\ N & 0 \end{smallmatrix}\right)$, then we have that $W : M_k(\Gamma_0(N)) \to M_k(\Gamma_0(N))$ and $W : S_k(\Gamma_0(N)) \to S_k(\Gamma_0(N))$.*

Proof. First note that if $\gamma = \left(\begin{smallmatrix} a & b \\ cN & d \end{smallmatrix}\right) \in \Gamma_0(N)$ then $\gamma' = \left(\begin{smallmatrix} d & -c \\ -bN & a \end{smallmatrix}\right) \in \Gamma_0(N)$ and we have that

$$\omega\gamma = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}\begin{pmatrix} a & b \\ cN & d \end{pmatrix} = \begin{pmatrix} -c\sqrt{N} & -d/\sqrt{N} \\ a\sqrt{N} & b\sqrt{N} \end{pmatrix} = \begin{pmatrix} d & -c \\ -bN & a \end{pmatrix}\begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix} = \gamma'\omega.$$

Thus if $f \in M_k(\Gamma_0(N))$ then, by modularity we have that

$$(Wf)|_k\gamma = f|_k\omega|_k\gamma = f|_k\omega\gamma = f|_k\gamma'\omega = f|_k\gamma'|_k\omega = f|_k\omega = Wf.$$

Thus $Wf \in M_k(\Gamma_0(N))$ also.

☐

Lemma 2.62 (The Fricke Involution Commutes with Hecke Operators)**.** *We have that $WT_n = T_n W$.*

Proof. Note that

$$WT_n f = n^{k/2-1} \sum_{\delta\in\Delta_n^N} f|_k\delta|_k\omega = n^{k/2-1} \sum_{\delta\in\Delta_n^N} f|_k\delta\omega.$$

Now for $\delta \in \Delta_n^N$ there exists $\gamma' \in \Gamma_0(N)$ and $\delta' \in \Delta_n^N$ such that $\delta\omega = \omega\gamma'\delta'$ and indeed this map $\delta \mapsto \delta'$ forms a bijection across elements of Δ_n^N. Thus,

$$WT_n f = n^{k/2-1} \sum_{\delta\in\Delta_n^N} f|_k\delta\omega = n^{k/2-1} \sum_{\delta'\in\Delta_n^N} f|_k\omega\gamma'\delta' = n^{k/2-1} \sum_{\delta'\in\Delta_n^N} f|_k\omega|_k\gamma'|_k\delta' = n^{k/2-1} \sum_{\delta'\in\Delta_n^N} (Wf)|_k\gamma'|_k\delta'.$$

Now since $Wf : M_k(\Gamma_0(N)) \to M_k(\Gamma_0(N))$ we know that Wf is modular, thus

$$WT_nf = n^{k/2-1} \sum_{\delta' \in \Delta_n^N} (Wf)|_k\gamma'|_k\delta' = n^{k/2-1} \sum_{\delta' \in \Delta_n^N} (Wf)|_k\delta' = T_nWf$$

exactly as desired. □

Lemma 2.63 (Hecke Eigenforms are Eigenforms of the Fricke Involution)**.** *If f is a Hecke eigenform then f is an eigenform of the Fricke involution. Specifically, $Wf = \pm f$.*

Proof. Now note that if f is a Hecke eigenform, then $g = f/c_1$ is a newform. Suppose $T_ng = \lambda_n g$ for all n. Now note

$$T_nWg = WT_ng = W\lambda_n g = \lambda_n Wg.$$

Thus Wg is a Hecke eigenform, and we know that $Wg = wh$ where h is a newform. Thus

$$wT_nh = T_nWg = \lambda_n Wg = w\lambda_n h \implies T_nh = \lambda_n h.$$

But since g and h are newforms with the same Hecke eigenvalues, by the multiplicity one theorem we know that $g = h$. Thus,

$$Wf = c_1Wg = c_1wh = c_1wg = wf$$

as desired. Now note that since

$$\omega^2 = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma_0(N),$$

by modularity we have that

$$f = f|_k\omega^2 = f|_k\omega|_k\omega = WWf = wWf = wwf = w^2 f.$$

Thus $w^2 = 1$ and so $w = \pm 1$. □

Theorem 2.64 (Functional Equations for L-Functions Associated with Hecke Eigenforms)**.** *For $f \in S_k$ where f is a Hecke eigenform, we can extend $L(f, s)$ to an entire function, and if*

$$\Lambda(f, s) = N^{s/2}\mathcal{M}(f(it))(s) = N^{s/2}\,(2\pi)^{-s}\,\Gamma(s)\,L(f, s)$$

then we have that

$$\Lambda(f, s) = \pm i^k \Lambda(f, k - s).$$

Proof. Recall that for all $\sigma > 0$, $\Gamma(s)$ is defined as

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt.$$

Now if we do the substitution $t = 2\pi nu$, then after some manipulation we have that

$$(2\pi n)^{-s}\,\Gamma(s) = \int_0^\infty e^{-2\pi nt}t^{s-1}\,dt.$$

Now if we take the sum over all $n \geq 1$ with the coefficients c_n.

$$(2\pi)^{-s}\,L(f, s)\,\Gamma(s) = \sum_{1\leq n} c_n\,(2\pi n)^{-s}\,\Gamma(s) = \sum_{1\leq n} c_n \int_0^\infty e^{-2\pi nt}t^{s-1}\,dt.$$

Now note that, using Hecke's bound and the substitution $u = 2\pi nt$ we have

$$\begin{aligned}\sum_{1\leq n} \int_0^\infty \left|c_n\,e^{-2\pi nt}t^{s-1}\right| dt &\ll \sum_{1\leq n} n^{k/2} \int_0^\infty e^{-2\pi nt}t^{\sigma-1}\,dt \\ &= (2\pi)^{-\sigma} \sum_{1\leq n} n^{-(\sigma-k/2)} \int_0^\infty e^{-u}u^{\sigma-1}\,du = \frac{\zeta(\sigma - k/2)\,\Gamma(\sigma)}{(2\pi)^\sigma}.\end{aligned}$$

For $\sigma > 1 + k/2$ we know this converges. So by Fubini-Tonelli, when $\sigma > 1 + k/2$, we can interchange the sum and the integral as needed.

$$(2\pi)^{-s}\, L(f,s)\,\Gamma(s) = \sum_{1\le n} c_n \int_0^\infty e^{-2\pi n t} t^{s-1}\, dt = \int_0^\infty \sum_{1\le n} c_n e^{-2\pi n t} t^{s-1}\, dt = \int_0^\infty f(it)\, t^{s-1}\, dt = \mathcal{M}(f(it))(s).$$

Now note that

$$(2\pi)^{-s}\, L(f,s)\,\Gamma(s) = \int_0^\infty f(it)\, t^{s-1}\, dt = \int_0^{1/\sqrt{N}} f(it)\, t^{s-1}\, dt + \int_{1/\sqrt{N}}^\infty f(it)\, t^{s-1}\, dt.$$

Now using the substitution $t = 1/(Nu)$ on the first integral we have

$$\begin{aligned}\int_0^{1/\sqrt{N}} f(it)\, t^{s-1}\, dt &= -N^{-s} \int_\infty^{1/\sqrt{N}} f\left(-\frac{1}{Niu}\right) u^{-s-1}\, du \\ &= N^{k/2-s} i^k \int_{1/\sqrt{N}}^\infty (Wf)(iu)\, u^{k-s-1}\, du = \pm N^{k/2-s} i^k \int_{1/\sqrt{N}}^\infty f(iu)\, u^{k-s-1}\, du.\end{aligned}$$

So we have that

$$(2\pi)^{-s}\, L(f,s)\,\Gamma(s) = \pm N^{k/2-s} i^k \int_{1/\sqrt{N}}^\infty f(it)\, t^{k-s-1}\, dt + \int_{1/\sqrt{N}}^\infty f(it)\, t^{s-1}\, dt.$$

Note that this integral is entire since $f(it)$ has exponential decay as $t \to \infty$ because f is a cusp form. Thus, we can extend $L(f,s)$ to an entire function. Returning to our line of thought, by replacing s with $k - s$ and multiplying through by $\pm N^{k/2-s} i^k$ we have

$$\pm N^{k/2-s} i^k\, (2\pi)^{-(k-s)}\, L(f,k-s)\,\Gamma(k-s) = i^{2k} \int_{1/\sqrt{N}}^\infty f(it)\, t^{s-1}\, dt \pm N^{k/2-s} i^k \int_{1/\sqrt{N}}^\infty f(it)\, t^{k-s-1}\, dt.$$

We know that k must be even since f is a cusp form, thus $i^{2k} = 1$. So we have that

$$\Lambda(f,s) = N^{s/2}\, (2\pi)^{-s}\, L(f,s)\,\Gamma(s) = \pm i^k N^{(k-s)/2} (2\pi)^{-(k-s)}\, L(f,k-s)\,\Gamma(k-s) = \pm i^k \Lambda(f,k-s)$$

as desired. □

Theorem 2.65 (Euler Product Expansion of L-Functions Associated with Newforms)**.** *If $f \in S_k^{new}(\Gamma_0(N))$ is a newform, then we have that*

$$f(z) = \sum_{1\le m} c_m e^{2\pi i m z} \implies L(f,s) = \prod_p \left(1 - \frac{c_p}{p^s} + \chi_0^N(p)\, \frac{p^{k-1}}{p^{2s}}\right)^{-1}.$$

Proof. Since f is a newform we know that f is a normalized simultaneous eigenform of all the Hecke operators T_n. Thus we know that the Fourier coefficients are multiplicative, so we have that

$$L(f,s) = \sum_{1\le n} \frac{c_n}{n^s} = \prod_p \sum_{0\le m} \frac{c_{p^m}}{p^{ms}} = \prod_p S_p \qquad \text{where} \qquad S_p = 1 + \sum_{1\le m} \frac{c_{p^m}}{p^{ms}}.$$

Now note that

$$\lambda_n f = T_n f = \sum_{0\le m} \sum_{d|(m,n)} \chi_0^N(d)\, d^{k-1} c_{mn/d^2} e^{2\pi i m z}.$$

Matching the $m = 1$ term we have that $\lambda_n c_1 = c_n$, and since $c_1 = 1$ we have that $\lambda_n = c_n$. Thus $c_n f = \lambda_n f = T_n f$ for all n. Note that it then follows that

$$c_p c_{p^m} f = T_p T_{p^m} f = \sum_{d|(p,p^m)} \chi_0^N(d)\, d^{k-1} T_{p^{m+1}/d^2} f = T_{p^{m+1}} f + \chi_0^N(p)\, p^{k-1} T_{p^{m-1}} f = c_{p^{m+1}} f + \chi_0^N(p)\, p^{k-1} c_{p^{m-1}} f.$$

Diving through by f we have that $c_p c_{p^m} = c_{p^{m+1}} + \chi_0^N(p)\, p^{k-1} c_{p^{m-1}}$. So we have that

$$\frac{c_p S_p}{p^s} = \frac{c_p}{p^s} + \sum_{1\le m} \frac{c_p c_{p^m}}{p^s p^{ms}} = \frac{c_p}{p^s} + \sum_{1\le m} \frac{c_{p^{m+1}} + \chi_0^N(p)\, p^{k-1} c_{p^{m-1}}}{p^{(m+1)s}} = \frac{c_p}{p^s} + \sum_{1\le m} \frac{c_{p^{m+1}}}{p^{(m+1)s}} + \frac{\chi_0^N(p)\, p^{k-1}}{p^{2s}} \sum_{1\le m} \frac{c_{p^{m-1}}}{p^{(m-1)s}}.$$

But now note that

$$\frac{c_p S_p}{p^s} = \left(\frac{c_p}{p^s} + \sum_{1 \le m} \frac{c_{p^{m+1}}}{p^{(m+1)s}} \right) + \frac{\chi_0^N(p)\, p^{k-1}}{p^{2s}} \sum_{1 \le m} \frac{c_{p^{m-1}}}{p^{(m-1)s}} = (S_p - 1) + \frac{\chi_0^N(p)\, p^{k-1}\, S_p}{p^{2s}}.$$

Rearranging we have that

$$S_p = \left(1 - \frac{c_p}{p^s} + \chi_0^N(p)\, \frac{p^{k-1}}{p^{2s}} \right)^{-1} \implies L(f,s) = \prod_p \left(1 - \frac{c_p}{p^s} + \chi_0^N(p)\, \frac{p^{k-1}}{p^{2s}} \right)^{-1}$$

as desired. □