Complex Analysis Qualifying Exam Prep

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Contents

1 Multivariable Calculus

1.1 Green's Theorem

Theorem 1.1.1. *Let C be a piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. Now if M and N are defined on an open region containing D then we have that*

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Remark. *Note that Green's theorem follows as a corollary to the generalized Stoke's theorem.*

(1) Example Problems: Green's Theorem

Example 1.1.2 (Fall 2023, Problem 1)**.** Use Green's theorem to evaluate the integral

$$\oint_C \sqrt{1+e^{x^2}}\,dx + 4xy\,dy$$

where C is the boundary of the triangle with vertices $(0,0), (1,0)$, and $(1,3)$ with the standard orientation.

Solution. Note that

$$\frac{\partial}{\partial x}(4xy) = 4y \qquad \text{and} \qquad \frac{\partial}{\partial y}\sqrt{1+e^{x^2}} = 0.$$

Thus by Green's theorem we have that

$$\oint_C \sqrt{1+e^{x^2}}\,dx + 4xy\,dy = \int_0^1 \int_0^{3x} 4y\,dy\,dx = \int_0^1 \left(2y^2\Big|_{y=0}^{3x}\right) dx = \int_0^1 18x^2\,dx = 6x^3\Big|_{x=0}^{1} = 6. \qquad \square$$

1.2 Jacobian Matrix: Change of Variables

Lemma 1.2.1. *Let D be a region bounded by a piecewise smooth, simple closed curve in the plane. Let f, g, and h be continuous functions, we have that*

$$\iint_D f(x,y)\,dx\,dy = \iint_D f(g(x',y'),h(x',y'))\cdot \det\begin{pmatrix}\partial g/\partial x' & \partial g/\partial y' \\ \partial h/\partial x' & \partial h/\partial y'\end{pmatrix} dx'\,dy'.$$

This generalizes to higher dimensions in the natural way.

(1) Example Problems: Jacobian Matrix

Example 1.2.2 (Spring 2021, Problem 5)**.** Let R be the parallelogram $(0,0),(1,1),(3,0)$, and $(2,-1)$. Evaluate

$$\iint_R (x+2y)^2 e^{x-y}\,dA.$$

Solution. We would like to apply the transformation

$$\begin{pmatrix}x' \\ y'\end{pmatrix} = \begin{pmatrix}1 & 2 \\ 1 & -1\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} \qquad \text{or alternatively} \qquad -\frac{1}{3}\begin{pmatrix}-1 & -2 \\ -1 & 1\end{pmatrix}\begin{pmatrix}x' \\ y'\end{pmatrix} = \begin{pmatrix}x \\ y\end{pmatrix}.$$

Thus we have

$$\iint_R (x+2y)^2 e^{x-y}\,dA = -\frac{1}{3}\int_0^3\int_0^3 (x')^2 e^{y'}\cdot\det\begin{pmatrix}-1 & -2 \\ -1 & 1\end{pmatrix} dx'\,dy' = \int_0^3\int_0^3 (x')^2 e^{y'}\,dx'\,dy'.$$

This is now a very straight-forward integral to evaluate.

$$\int_0^3\int_0^3 (x')^2 e^{y'}\,dx'\,dy' = \int_0^3 e^{y'}\left(\left.\frac{x^3}{3}\right|_{x=0}^{3}\right) dy' = 9\int_0^3 e^{y'}\,dy' = 9\left(\left.e^{y'}\right|_{y=0}^{3}\right) = 9\,(e^3-1). \qquad \square$$

2 Complex Analysis

2.1 Harmonic Functions

Definition 2.1.1. *A function f is harmonic if it satisfies the following equation*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Lemma 2.1.2. *If a function $u(x, y)$ is harmonic, then there exists a harmonic conjugate of u denoted $v(x, y)$ such that $f(z) = u(x, y) + i \cdot v(x, y)$ is holomorphic in $z = x + i \cdot y$.*

Corollary 2.1.3. *A function u is harmonic if and only if there exists a holomorphic function f such that $u = \mathfrak{Re}(f)$.*

(1) Example Problems: Harmonic Functions

Example 2.1.4 (Fall 2022, Problem 1)**.** Show that $u(x, y) = \ln\left(x^2 + y^2\right)$ is a harmonic function in $\mathbb{C} \setminus \{0\}$. Find a conjugate harmonic function of $u(x, y)$ in $\mathbb{C} \setminus \{x : x \leq 0\}$. Show that it does not have a conjugate harmonic function in $\mathbb{C} \setminus \{0\}$.

Solution. Recall that a harmonic function f satisfies the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial^2 y} = 0.$$

So, note that

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{2(x^2 + y^2) - 2x(2x)}{(x^2 + y^2)^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}.$$

And similarly,

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{2(x^2 + y^2) - 2y(2y)}{(x^2 + y^2)^2} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}.$$

So trivially we have that $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ wherever this function is defined. Noting that $u(z) = u(x, y) = \ln\bigl(x^2 + y^2\bigr) = 2 \ln |z|$, it is clear that this function is defined on $\mathbb{C} \setminus \{0\}$.

Since u is harmonic we know there exists a holomorphic function f such that $u = \mathfrak{Re}(f)$. There is the obvious choice:

$$f(z) = 2\,\mathrm{Log}(z) = 2 \ln |z| + 2i\,\mathrm{Arg}(z)).$$

Indeed $u = \mathfrak{Re}(f)$. Taking the imaginary part yields the harmonic conjugate

$$v(x, y) = 2\,\mathrm{Arg}(x + iy) = 2 \arctan(y/x).$$

Note that this function is continuous on the right half plane but not $\mathbb{C} \setminus \{0\}$; hence this harmonic conjugate is valid for the right half plane and not $\mathbb{C} \setminus \{0\}$.

(Incomplete still)

2.2 Power Series

Definition 2.2.1. *Given a power series $\sum_{1 \le n} a_n z^n$ there exists a number $0 \le R \le \infty$ such that for all $|z| < R$ the series converges absolutely, and for all $|z| > R$ the series diverges. This R is the radius of convergence of the power series $\sum_{1 \le n} a_n z^n$.*

Theorem 2.2.2. *We have that the radius of convergence satisfies*

$$1/R = \limsup_{n \to \infty} |a_n|^{1/n}.$$

(2) Example Problems: Power Series

Example 2.2.3 (Spring 2024, Problem 2)**.** Let

$$F(z) = \sum_{n=1}^{\infty} \frac{z^n}{1 - z^n}.$$

Find its power series $\sum_{k=1}^{\infty} a_k z^k$ and find its radius of convergence.

Solution. Working over $\mathbb{C}[[z]]$, finding the power series becomes trivial.

$$F(z) = \sum_{1 \le n} \frac{z^n}{1 - z^n} = \sum_{1 \le n} \sum_{1 \le k} z^{kn} = \sum_{1 \le n} \sum_{d \mid n} z^n = \sum_{1 \le n} \sigma_0(n)\, z^n.$$

Now note that $2 \le \sigma_0(n) \le n$ for all $n > 1$ (trivially). Thus we have that

$$\limsup_{n \to \infty} 2^{1/n} \le \limsup_{n \to \infty} |a_n|^{1/n} \le \limsup_{n \to \infty} n^{1/n}.$$

Note that

$$\left(\lim_{n \to \infty} 2^{1/n} = \lim_{n \to \infty} n^{1/n} = 1 \right) \implies \left(\limsup_{n \to \infty} 2^{1/n} = \limsup_{n \to \infty} n^{1/n} = 1 \right).$$

Thus $1/R = \limsup_{n \to \infty} |a_n|^{1/n} = 1$ and so $R = 1$.

Example 2.2.4 (Stein-Shakarchi, Chapter 1 Problem 16abc)**.** Find the radius of convergence for $\sum_{0 \le n} a_n z^n$ when

- $a_n = \log^2 n$
- $a_n = n!$
- $a_n = n^2/(4^n + 3n)$

Solution.

- Starting with $a_n = \log^2 n$. Note that

$$\lim_{n \to \infty} (\log^2 n)^{1/n} = \lim_{n \to \infty} (\log n)^{2/n} = \exp\left(2 \lim_{n \to \infty} \frac{\log \log n}{n} \right) = \exp\left(2 \lim_{n \to \infty} \frac{1}{n \ln n} \right) = 1.$$

  Thus, $1/R = \limsup_{n \to \infty} (\log^2 n)^{1/n} = 1$ and so $R = 1$.

- Now working with $a_n = n!$, we note that by Sterling's approximation $\log n! = n \log n - n + O(\log n)$. So we have

$$\lim_{n \to \infty} (n!)^{1/n} = \lim_{n \to \infty} \exp\left(\frac{\log n!}{n} \right) = \lim_{n \to \infty} \exp(\log n - 1) = \infty.$$

  Thus $1/R = \limsup_{n \to \infty} (n!)^{1/n} = \infty$ and so $R = 0$.

- Now working with $a_n = n^2/(4^n + 3n)$, note that for $n \ge 0$ we have $4^n \le 4^n + 3n \le 2 \cdot 4^n$. Thus it follows,

$$\frac{1}{4} = \frac{1}{4} \lim_{n \to \infty} (n/2)^{2/n} \le \lim_{n \to \infty} \left(\frac{n^2}{4^n + 3n} \right)^{1/n} \le \frac{1}{4} \lim_{n \to \infty} n^{2/n} = \frac{1}{4}.$$

  Thus $1/R = \limsup_{n \to \infty} (n^2/(4^n + 3n))^{1/n} = 1/4$ and so $R = 4$.

Example 2.2.5 (Spring 2024, Problem 5)**.** Let

$$f(z)=\sum_{n=1}^{\infty}\frac{1}{n^8}z^{3^n}$$

which has convergence radius 1 . (Thus $f(z)$ is a well defined holomorphic function over the unit disk $\Delta:=\{z\in\mathbb{C}: |z|<1\}$.)

- Prove that $f(z)$ does not admit a holomorphic extension to a neighborhood of 1 in $\mathbb{C}$. Namely, there do not exist a neighborhood U of 1 in the complex plane $\mathbb{C}$ and a holomorphic function g over U such that $f|_{Un\Delta}=g|_{U\cap\Delta}$.
- Further show that the unit disk is the natural defining domain of $f(z)$. Namely, there do not exit a domain Ω strictly larger than the unit disk and a holomorphic function F defined over Ω such that the restriction of F to the unit disk is $f(z)$.

Solution. Suppose there exists a holomorphic extension of f to a neighborhood of 1. Then we know that f' is also holomorphic in a neighborhood of 1 as well. Note that

$$f'(z)=\sum_{n=1}^{\infty}\frac{3^n}{n^8}\,z^{3^n-1}.$$

Now let $z=re^{i\theta}$ where $\theta=2\pi k/3^N$ where $0\leq k<3^N$. Thus

$$f'(z)=\sum_{n=1}^{\infty}\frac{3^n}{n^8}\left(r^{3^n-1}\exp\bigl(2\pi ik(3^n-1)/3^N\bigr)\right)=\exp\bigl(-2\pi ik/3^N\bigr)\sum_{n=1}^{\infty}\frac{3^n}{n^8}\left(r^{3^n-1}\exp\bigl(2\pi ik\cdot 3^{n-N}\bigr)\right).$$

But now we can rearrange as follows:

$$\exp\bigl(2\pi ik/3^N\bigr)\,f'(z)=\sum_{n=1}^{N-1}\frac{3^n}{n^8}\left(r^{3^n-1}\exp\bigl(2\pi ik\cdot 3^{n-N}\bigr)\right)+\sum_{n=N}^{\infty}\frac{3^n}{n^8}\,r^{3^n-1}.$$

Now note that

$$\lim_{r\to1^-}\sum_{n=N}^{\infty}\frac{3^n}{n^8}r^{3^n-1}=\infty\qquad\text{and}\qquad\left|\sum_{n=1}^{N-1}\frac{3^n}{n^8}\left(r^{3^n-1}\exp\bigl(2\pi ik\cdot 3^{n-N}\bigr)\right)\right|<\infty.$$

Thus by reverse triangle inequality, for r sufficiently large we have that

$$\sum_{n=N}^{\infty}\frac{3^n}{n^8}\,r^{3^n-1}-\left|\sum_{n=1}^{N-1}\frac{3^n}{n^8}\left(r^{3^n-1}\exp\bigl(2\pi ik\cdot 3^{n-N}\bigr)\right)\right|\leq|f'(z)|.$$

But note that the LHS goes to infinity as r goes to 1. Thus $\lim_{r\to1^-}|f'(z)|=\infty$ as well.

Now note that if we choose $k=0$ and $N=0$, then $z=r$. So $\lim_{r\to1^-}|f'(z)|=\infty$ implies that f' can not be holomorphic in any neighborhood of 1, because f' is not holomorphic at 1. Contradiction, thus f can not be holomorphic in a neighborhood of 1.

Suppose there exists $\Omega\supset\mathbb{D}$ such that there exists a holomorphic extension of f to Ω. Then we know that f' is also holomorphic on Ω as well. Because our choices of z are dense on $\partial\mathbb{D}$, if we choose any $\Omega\supset\mathbb{D}$, it must contain some z in our dense set. So $\lim_{r\to1^-}|f'(z)|=\infty$ implies that f' can not be holomorphic in any neighborhood of z, because f' is not holomorphic at z. Thus f' can not be holomorphic on Ω. Contradiction, thus f can not be holomorphic on Ω.

Example 2.2.6 (Stein-Shakarchi, Chapter 1 Problem 19)**.** Prove the following:

- The power series $\sum nz^n$ does not converge at any point on the unit circle.
- The power series $\sum z^n/n^2$ converges at every point on the unit circle.
- The power series $\sum z^n/n$ converges at every point on the unit circle except for $z = 1$.

Solution.

- In the case of $\sum nz^n$ note that when $|z| = 1$ we have $\lim_{n\to\infty} |nz^n| = \lim_{n\to\infty} n|z|^n = \lim_{n\to\infty} n = \infty$. So the sum can not converge.
- Now in the case of $\sum z^n/n^2$ note that when $|z| = 1$ we have

$$\sum \left|z^n/n^2\right| = \sum |z|^n/n^2 = \sum n^{-2} = \pi^2/6.$$

  Since the sum converges absolutely, the original series must also converge.

- Note that for all $z \neq 1$ and $N \in \mathbb{N}$ that

$$\left|\sum_{n=1}^{N} z^n\right| = \left|\frac{1 - z^N}{1 - z}\right| \leq \frac{2}{|1 - z|}.$$

  So it follows from summation by parts that

$$\sum_{n=1}^{N} \frac{z^n}{n} = \frac{1}{N}\sum_{n=1}^{N} z^n + \sum_{k=1}^{N-1}\left(\frac{1}{k^2 + k}\right)\sum_{n=1}^{k} z^n.$$

  Now note that

$$0 \leq \left|\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} z^n\right| \leq \lim_{N\to\infty} \frac{1}{N}\left|\sum_{n=1}^{N} z^n\right| \leq \lim_{N\to\infty} \frac{2}{N\,|1 - z|} = 0 \qquad \Longrightarrow \qquad \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} z^n = 0.$$

  Thus we have that

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = \sum_{k=1}^{\infty}\left(\frac{1}{k^2 + k}\right)\sum_{n=1}^{k} z^n.$$

  But note that testing absolute convergence of the right hand side we have

$$\sum_{k=1}^{\infty}\left|\left(\frac{1}{k^2 + k}\right)\sum_{n=1}^{k} z^n\right| \leq \frac{2}{|1 - z|}\sum_{k=1}^{\infty}\frac{1}{k^2 + k} \leq \frac{2}{|1 - z|}\sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{3\,|1 - z|}.$$

  Since the series converges absolutely the original series converges for all $z \neq 1$.

2.3 Rouche's Theorem

Theorem 2.3.1. *For any two holomorphic functions f and g on some region K with closed contour ∂K, it follows that if $|g(z)| < |f(z)|$ on ∂K then f and $f + g$ have the same number of zeros inside K.*

(6) Example Problems: Rouche's Theorem

Example 2.3.2 (Spring 2024, Problem 3)**.** Find the number of roots of $z^4 - 6z + 3 = 0$ such that $1 < |z| < 2$.

Solution. Note that when $|z| = 2$ we have that

$$|-6z| = 6|z| = 12 < 13 = \left||z|^4 - 3\right| \leq \left|z^4 + 3\right|.$$

Thus by Rouche's theorem, $z^4 + 3 = 0$ and $z^4 - 6z + 3 = 0$ have the same number of roots with $|z| < 2$. Noting that

$$z^4 + 3 = (z^2 + i\sqrt{3})(z^2 - i\sqrt{3}) = (z - \sqrt[4]{3})(z + \sqrt[4]{3})(z - i\sqrt[4]{3})(z + i\sqrt[4]{3})$$

and that $\left|\sqrt[4]{3}\right| < 2$ gives us that $z^4 - 6z + 3$ has four roots such that $|z| < 2$.

Note that when $|z| = 1$ we have that

$$\left|z^4\right| = |z|^4 = 1 < 3 = |6|z| - 3| \leq |-6z + 3|.$$

Thus by Rouche's theorem $-6z + 3 = 0$ and $z^4 - 6z + 3 = 0$ have the same number of roots with $|z| < 1$. Noting that

$$-6z + 3 = -6(z - 1/2)$$

and that $|1/2| < 1$ gives us that $z^4 - 6z + 3$ has one root such that $|z| < 1$.

Thus there are three roots of $z^4 - 6z + 3 = 0$ such that $1 < |z| < |2|$.

Example 2.3.3 (Fall 2021, Problem 1)**.** Fix $0 < R < \pi/2$. Prove that for sufficiently large n the polynomial

$$P_n(z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots + \frac{z^{2n}}{(2n)!} = 0$$

has no roots such that $|z| < R$.

Solution. Note that

$$\cos(iz) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots = \frac{1}{2}\left(e^z + e^{-z}\right).$$

Now fix $0 < R < \pi/2$ and let $\varepsilon > 0$ be such that $R + \varepsilon < \pi/2$ also. Since $\cos(iz)$ has zeros at $\pm\pi i/2, \pm 3\pi i/2, \ldots$ there exists M such that $0 < M < |\cos(iz)|$ for $|z| < R + \varepsilon$. Now note that

$$\begin{aligned} |P_n(z) - \cos(iz)| &= \left| -\frac{z^{2n+2}}{(2n+2)!} - \frac{z^{2n+4}}{(2n+4)!} - \frac{z^{2n+6}}{(2n+6)!} + \cdots \right| \\ &\leq \frac{|z|^{2n}}{(2n)!}\left(\frac{|z|^2}{(2n+1)(2n+2)} + \frac{|z|^4}{(2n+1)\cdots(2n+4)} + \frac{|z|^6}{(2n+1)\cdots(2n+6)} + \cdots\right) \\ &\leq \frac{|z|^{2n}}{(2n)!}\left(1 + \frac{|z|^2}{2!} + \frac{|z|^4}{4!} + \frac{|z|^6}{6!} + \cdots\right) = \frac{|z|^{2n}}{(2n)!}\left(\frac{\exp(|z|) + \exp(-|z|)}{2}\right) \end{aligned}$$

Thus if $|z| = R$ then we have

$$|P_n(z) - \cos(iz)| \leq \frac{R^{2n}}{(2n)!}\left(\frac{e^R + e^{-R}}{2}\right).$$

Noting that the right hand side of the above goes to 0 pointwise in terms of R as $n \to \infty$, there exists some N such that for all $n > N$ and $|z| = R$ we have

$$|P_n(z) - \cos(iz)| \leq \frac{R^{2n}}{(2n)!}\left(\frac{e^R + e^{-R}}{2}\right) < M < |\cos(iz)|.$$

Now by Rouche's theorem we know that $P_n(z) = 0$ and $\cos(iz) = 0$ have the same number of roots such that $|z| < R$.

Thus there exists sufficiently large n such that $P_n(z) = 0$ has no roots such that $|z| < R$.

Example 2.3.4 (Spring 2021, Problem 1)**.** Prove that all five roots of $2z^5 + 8z - 1 = 0$ are such that $|z| < 2$ but only one root is such that $|z| < 1$.

Solution. Note that when $|z| = 2$ we have that

$$|8z| = 8|z| = 16 < 31 = \left|2|z|^5 - 1\right| \leq \left|2z^5 - 1\right|.$$

Thus by Rouche's theorem, $2z^5 - 1 = 0$ and $2z^5 + 8z - 1 = 0$ have the same number of roots with $|z| < 2$. Note that

$$2z^5 - 1 = (z - \sqrt[5]{1/2})(z - e^{2\pi i/5}\sqrt[5]{1/2})(z - e^{4\pi i/5}\sqrt[5]{1/2})(z - e^{6\pi i/5}\sqrt[5]{1/2})(z - e^{8\pi i/5}\sqrt[5]{1/2})$$

and that $\left|\sqrt[5]{1/2}\right| < 2$ gives us that $2z^5 + 8z - 1$ has five roots such that $|z| < 2$.

Note that when $|z| = 1$ we have that

$$\left|2z^5\right| = 2|z|^5 = 2 < 7 = |8|z| - 1| \leq |8z - 1|.$$

Thus by Rouche's theorem, $8z - 1 = 0$ and $2z^5 + 8z - 1 = 0$ have the same number of roots with $|z| < 1$. Noting that

$$8z - 1 = 8(z - 1/8)$$

and that $|1/8| < 1$ gives us that $2z^5 + 8z - 1$ has one root such that $|z| < 1$.

Example 2.3.5 (Spring 2023, Problem 2)**.** Fix $\lambda \in \mathbb{C}$ such that it is purely imaginary. Prove that $z = \lambda - e^{z^2}/3$ has exactly one solution in the strip $\mathbb{S} = \{z \in \mathbb{C} : |\mathfrak{Re}(z)| < 1\}$.

Solution. Let $x = \mathfrak{Re}(z)$ and $y = \mathfrak{Im}(z)$ and note that for $z \in \mathbb{S}$ we have that $|x| \leq 1$. We have that

$$\left|e^{z^2}/3\right| = e^{\mathfrak{Re}(z^2)}/3 = e^{x^2 - y^2}/3 = e^{x^2}e^{-y^2}/3 \leq e^{x^2}/3 \leq e/3 < 1.$$

Now let R_r be the open rectangle with opposite corners $(-1, \lambda - r)$ and $(1, \lambda + r)$. Now note that for $r \geq 1$ for any $z \in \partial R_r$ we have that $|z - \lambda| \geq 1$. Thus for all $r \geq 1$ and $z \in \partial R_r$ we have

$$\left|e^{z^2}/3\right| < 1 \leq |z - \lambda|.$$

Thus by Rouche's theorem, $z - \lambda = 0$ and $z - \lambda + e^{z^2}/3 = 0$ have the same number of roots in R_r. Thus $z - \lambda + e^{z^2}/3$ has one root in R_r for all $r \geq 1$. Noting that

$$R_1 \subset R_2 \subset R_3 \subset \ldots \subset \mathbb{S} \qquad \text{and} \qquad \bigcup_{1 \leq r} R_r = \mathbb{S}$$

gives us that $z - \lambda + e^{z^2}/3 = 0$ has one solution in $\mathbb{S}$.

Example 2.3.6 (Spring 2022, Problem 2)**.** Show that $2 + z^2 - e^{iz} = 0$ has exactly one solution in the upper-half plane.

Solution. Let $x = \mathfrak{Re}(z)$ and $y = \mathfrak{Im}(z)$ and note that for $z \in \mathcal{H}$ we have that $y > 0$. Note that

$$\left|-e^{iz}\right| = e^{-y} < 1.$$

Now let R_r be the open rectangle with opposite corners $(-r, 0)$ and (r, r). Now for any $z \in \mathbb{C}$ note that

$$\left|2 + x^2 - y^2\right| \leq \sqrt{(2 + x^2 - y^2)^2 + (2xy)^2} = \left|2 + z^2\right| \qquad \text{and that} \qquad \left|2 - |z|^2\right| \leq \left|2 + z^2\right|.$$

Now let $r \geq 2$. Then for any point $z \in \partial R_r$ on the bottom edge we have $-r \leq x \leq r$ and $y = 0$; so

$$2 \leq 2 + x^2 = \left|2 + x^2 - 0^2\right| \leq \left|2 + z^2\right|.$$

Now for any point $z \in \partial R_r$ on any other edge we have $|z| \geq r$; so

$$2 \leq r^2 - 2 \leq |z|^2 - 2 = \left|2 - |z|^2\right| \leq \left|2 + z^2\right|.$$

Thus when $r \geq 2$ for every $z \in R_r$ we have that $2 \leq \left|2 + z^2\right|$ and by extension $\left|-e^{iz}\right| < 1 < 2 \leq \left|2 + z^2\right|$. So by Rouche's theorem we have that $2 + z^2 = 0$ and $2 + z^2 - e^{iz} = 0$ have the same number of roots in R_r. Noting that $2 + z^2 = (z - i\sqrt{2})(z + i\sqrt{2})$, we have that $2 + z^2 - e^{iz}$ has one root in R_r for all $r \geq 2$. Noting that

$$R_2 \subset R_3 \subset R_4 \subset \ldots \subset \mathcal{H} \qquad \text{and} \qquad \bigcup_{2 \leq r} R_r = \mathcal{H}$$

gives us that $2 + z^2 - e^{iz} = 0$ has one solution in $\mathcal{H}$.

Example 2.3.7 (Fall 2020, Problem 2)**.** Prove that if $1 < a < \infty$ is a real number, then $f_a(z) = z + a - e^z$ has only one zero in the left-half plane and that the zero is real.

Solution. Now let $x = \mathbb{R}(z)$ and $y = \mathfrak{Re}(z)$ and note that for $z \in i\mathcal{H}$ we have that $x < 0$. Note that

$$|-e^z| = -e^x < 1.$$

Now let R_r be the open rectangle with opposite corners $(-r, -r)$ and $(0, r)$. Now note that by simple geometric reasoning for any $r \geq 2a$ and $z \in R_r$ we have that $|z + a| \geq a > 1$. Thus for all $r \geq 2a$ and $z \in R_r$ we have that

$$|-e^z| < 1 < a \leq |z + a|.$$

Thus by Rouche's theorem, $z + a = 0$ and $z + a - e^z = 0$ have the same number of roots in R_r. Thus $z + a - e^z$ has one root in R_r for all $r \geq 2a$. Noting that

$$R_{\lceil 2a \rceil} \subset R_{\lceil 2a \rceil + 1} \subset R_{\lceil 2a \rceil + 2} \subset \ldots \subset i\mathcal{H} \qquad \text{and} \qquad \bigcup_{\lceil 2a \rceil \leq r} R_r = i\mathcal{H}$$

gives us that $z + a - e^z = 0$ has one solution in $i\mathcal{H}$. Note that

$$f_a(0) = 0 + a - e^0 = a - 1 > 0 \qquad \text{and} \qquad f_a(-a) = -a + a - e^{-a} = -e^{-a} < 0.$$

Thus by the intermediate value theorem there exists at least one real zero of f_a on the interval $(-a, 0)$.

Since there exists one solution to $z + a - e^z = 0$ in $i\mathcal{H} \supset (-a, 0)$ and there exists at least one solution to $z + a - e^z = 0$ on the interval $(-a, 0)$, it follows that there exists one solution to $z + a - e^z = 0$ in $i\mathcal{H}$ and it is real.

2.4 Residue Theorem

Definition 2.4.1. *We define the residue of as follows. Given a function f holomorphic in a neighborhood of a point z_0, we can write the Laurent expansion*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(z-z_0)^n \qquad \text{and then} \qquad \operatorname{Res}(f, z_0) = a_{-1}.$$

Lemma 2.4.2. *If f has an order n pole at z_0 then*

$$\operatorname{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z).$$

Theorem 2.4.3. *Let f be holomorphic on an open set containing a closed contour C and let P be the set points inside C which are poles of f. Then we have that*

$$\oint_C f\, dz = 2\pi i \sum_{z_0\in P} \operatorname{Res}(f, z_0)$$

(8) Example Problems: Residue Theorem

Example 2.4.4 (Spring 2024, Problem 1)**.** For $a \neq 0$ evaluate

$$\int_0^\pi \tan(t+ai)\, dt.$$

Solution. Note that

$$\int_0^\pi \tan(t+ai)\, dt = \int_0^\pi \frac{\sin(t+ai)}{\cos(t+ai)}\, dt.$$

Recall the Weierstrass factorization of $\cos z$

$$\cos z = \prod_{k\neq 0}\left(1 - \frac{2z}{\pi(2k-1)}\right).$$

From this it follows that the poles of $\tan(t+ai)$ are of order 1 and are located at $\pm k\pi - ai$ where k is odd. Now for $a > 0$ we define the contour C as the counterclockwise orientation of boundary of the rectangle with opposite vertices $(0,0)$ and $(\pi, -R)$ where $0 < a < R$. Now we evaluate using the residue theorem

$$\oint_C \tan(z+ai)\, dz = 2\pi i \operatorname{Res}(f, \pi/2 - ai) = 2\pi i \lim_{z\to(\pi/2-ai)} (z - \pi/2 + ai)\tan(z+ai).$$

Note that this is an indeterminate form as

$$\lim_{z\to(\pi/2-ai)} (z-\pi/2+ai)\tan(z+ai) = \lim_{z\to(\pi/2-ai)} \frac{(z-\pi/2+ai)\sin(z+ai)}{\cos(z+ai)}$$

with the denominator and numerator both going to 0. We apply L'Hopital's rule to remedy this

$$= \lim_{z\to(\pi/2-ai)} \frac{(z-\pi/2+ai)\cos(z+ai) + \sin(z+ai)}{-\sin(z+ai)} = \frac{0\cdot 0+1}{-1} = -1.$$

Thus

$$\oint_C \tan(z+ai)\, dz = -2\pi i.$$

Now note that $\tan(z) = \tan(\pi + z)$. So, when $0 < a < R$ we have

$$\begin{aligned}-2\pi i = \oint_C \tan(z+ai)\, dz &= \int_0^{-R} \tan(ti+ai)\, dt + \int_0^\pi \tan(t - Ri + ai)\, dt - \int_0^{-R} \tan(\pi + ti + ai)\, dt - \int_0^\pi \tan(t+ai)\, dt \\ &= \int_0^\pi \tan(t-Ri+ai)\, dt - \int_0^\pi \tan(t+ai)\, dt.\end{aligned}$$

Thus when $a > 0$ and $R = 2a$ we have by algebraic manipulation

$$\begin{aligned}
-2\pi i &= \int_0^{\pi} \tan(t - ai)\,dt - \int_0^{\pi} \tan(t + ai)\,dt \\
&= -\int_0^{-\pi} \tan(-u - ai)\,du - \int_0^{\pi} \tan(t + ai)\,dt \\
&= \int_0^{-\pi} \tan(u + ai)\,du - \int_0^{\pi} \tan(t + ai)\,dt \\
&= -\int_{-\pi}^{0} \tan(u + ai)\,du - \int_0^{\pi} \tan(t + ai)\,dt = -\int_0^{\pi} \tan(t - \pi + ai)\,dt - \int_0^{\pi} \tan(t + ai)\,dt.
\end{aligned}$$

Now note that $\tan(z) = \tan(\pi + z)$. So

$$-2\pi i = -2\int_0^{\pi} \tan(t + ai)\,dz \implies \int_0^{\pi} \tan(t + ai)\,dt = \pi i.$$

Now for $b < 0$ and $a = -b$ we have that

$$\int_0^{\pi} \tan(t - bi)\,dt = \int_0^{\pi} \tan(t + ai)\,dt = \pi i.$$

But we have that

$$\pi i = \int_0^{\pi} \tan(t - bi) = -\int_0^{-\pi} \tan(-u - bi)\,du = -\int_{-\pi}^{0} \tan(u + bi)\,du = -\int_0^{\pi} \tan(t - \pi + bi)\,dt.$$

Now note that $\tan(z) = \tan(\pi + z)$. So

$$\int_0^{\pi} \tan(t + bi) = -\pi i.$$

Example 2.4.5 (Fall 2023, Problem 2)**.** Assume $\xi > 0$ and compute

$$\int_{\mathbb{R}} \frac{\cos(2\pi x\xi)}{x^2 + 1}\,dx.$$

Solution. Trivially the poles of

$$\frac{e^{2\pi iz\xi}}{z^2 + 1}$$

are located at $\pm i$ and are each of order 1. Let C be the counterclockwise orientation of the radius R upper-half semicircle centered at 0. Now for $R > 1$ we have by the residue theorem that

$$\oint_C \frac{e^{2\pi iz\xi}}{z^2 + 1}\,dz = 2\pi i \operatorname{Res}(f, i) = 2\pi i \lim_{z \to i} \frac{e^{2\pi iz\xi}}{x + i} = \pi e^{-2\pi\xi}.$$

Now let γ_1 be the segment of C along the real axis and let γ_2 be the semicircular part, preserving the orientation of both contours from C. Now note that because sin is odd we have

$$\int_{\gamma_1} \frac{e^{2\pi iz\xi}}{z^2 + 1}\,dz = \int_{-R}^{R} \frac{\cos(2\pi z\xi) + i\sin(2\pi z\xi)}{z^2 + 1}\,dz = \int_{-R}^{R} \frac{\cos(2\pi z\xi)}{z^2 + 1}\,dz.$$

Now note that

$$0 \le \left| \int_{\gamma_2} \frac{e^{2\pi iz\xi}}{z^2 + 1}\,dz \right| = \left| iR\int_0^{\pi} \frac{\exp\big(2\pi iRe^{i\theta}\xi\big)}{R^2e^{2i\theta} + 1}\,e^{i\theta}\,d\theta \right| \le R\int_0^{\pi} \left| \frac{\exp\big(2\pi iRe^{i\theta}\xi\big)}{R^2e^{2i\theta} + 1} \right| d\theta.$$

Now note that for $R > 1$ and $0 \le \theta \le \pi$ we have

$$\big|\exp\big(2\pi iRe^{i\theta}\xi\big)\big| = \exp(-2\pi\xi R\sin\theta) \le 1 \qquad \text{and that} \qquad R^2 - 1 \le \big|R^2e^{2i\theta} + 1\big|.$$

Thus we have that

$$0 \le \left| \int_{\gamma_2} \frac{e^{2\pi iz\xi}}{z^2 + 1}\,dz \right| \le R\int_0^{\pi} \left| \frac{\exp\big(2\pi iRe^{i\theta}\xi\big)}{R^2e^{2i\theta} + 1} \right| d\theta \le R\int_0^{\pi} \frac{d\theta}{R^2 - 1} = \frac{\pi R}{R^2 - 1}.$$

Thus by the squeeze theorem, in the limit $R \to \infty$ we have that the integral over γ_2 vanishes. So,

$$\pi e^{-2\pi\xi} = \lim_{R\to\infty} \oint_C \frac{e^{2\pi i z\xi}}{z^2+1}\,dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos(2\pi z\xi)}{z^2}\,dz + \lim_{R\to\infty} \int_{\gamma_2} \frac{e^{2\pi i z\xi}}{z^2+1}\,dz = \int_{-\infty}^{\infty} \frac{\cos(2\pi z\xi)}{z^2+1}\,dz.$$

Example 2.4.6 (Spring 2023, Problem 1)**.** Let $a, b > 0$ such that $a \neq b$; compute the integral

$$\int_{\mathbb{R}} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx.$$

Solution. Trivially the poles of

$$\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$$

are located at $\pm ia$ and $\pm ib$ and they all have order 1. Let C be the counterclockwise orientation of the radius R upper-half semicircle centered at 0. Now for $R > \max(a, b)$ we have by the residue theorem that

$$\begin{aligned}\oint_C \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz &= 2\pi i \operatorname{Res}(f, ia) + 2\pi i \operatorname{Res}(f, ib) \\ &= 2\pi i \lim_{z\to ia} \frac{e^{iz}}{(z+ia)(z^2+b^2)} + 2\pi i \lim_{z\to ib} \frac{e^{iz}}{(z^2+a^2)(z+ib)} \\ &= \frac{\pi e^{-a}}{a\,(b^2-a^2)} + \frac{\pi\, e^{-b}}{b\,(a^2-b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right).\end{aligned}$$

Now let γ_1 be the segment of C along the real axis and let γ_2 be the semicircular part, preserving the orientation of both contours from C. Now note that because sin is odd we have

$$\int_{\gamma_1} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz = \int_{-R}^{R} \frac{\cos z + i\sin z}{(z^2+a^2)(z^2+b^2)}\,dz = \int_{-R}^{R} \frac{\cos z}{(z^2+a^2)(z^2+b^2)}\,dz.$$

Now note that

$$0 \le \left|\int_{\gamma_2} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz\right| = \left| iR\int_0^{\pi} \frac{\exp\big(iRe^{i\theta}\big)}{(R^2e^{2i\theta}+a^2)(R^2e^{2i\theta}+b^2)}\, e^{i\theta}\,d\theta\right| \le R\int_0^{\pi} \left|\frac{\exp\big(iRe^{i\theta}\big)}{(R^2e^{2i\theta}+a^2)(R^2e^{2i\theta}+b^2)}\right| d\theta.$$

Now note that for $R > \max(a, b)$ and $0 \le \theta \le \pi$ we have

$$\left|\exp\big(iRe^{i\theta}\big)\right| = \exp(-R\sin\theta) \le 1 \qquad \text{and that} \qquad (R^2-a^2)(R^2-b^2) \le \left|(R^2e^{2i\theta}+a^2)(R^2e^{2i\theta}+b^2)\right|.$$

Thus we have that

$$\begin{aligned}0 \le \left|\int_{\gamma_2} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz\right| &\le R\int_0^{\pi} \left|\frac{\exp\big(iRe^{i\theta}\big)}{(R^2e^{2i\theta}+a^2)(R^2e^{2i\theta}+b^2)}\right| d\theta \\ &\le R\int_0^{\pi} \frac{d\theta}{(R^2-a^2)(R^2-b^2)} = \frac{\pi R}{(R^2-a^2)(R^2-b^2)}.\end{aligned}$$

Thus by the squeeze theorem, in the limit $R \to \infty$ we have that the integral over γ_2 vanishes. So,

$$\begin{aligned}\frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) &= \lim_{R\to\infty} \oint_C \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz \\ &= \lim_{R\to\infty} \int_{\gamma_1} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz + \lim_{R\to\infty} \int_{\gamma_2} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz = \int_{-\infty}^{\infty} \frac{\cos z}{(z^2+a^2)(z^2+b^2)}\,dz.\end{aligned}$$

Example 2.4.7 (Fall 2022, Problem 2)**.** Evaluate the integral

$$\int_{\mathbb{R}} \frac{x^2}{x^4+1}\,dx.$$

Solution. Note that
$$z^4+1=(z-e^{\pi i/4})(z+e^{\pi i/4})(z-e^{3\pi i}/4)(z+e^{3\pi i}/4).$$
Thus $z^2/(z^4+1)$ has poles at $e^{\pi i/4}$, $e^{3\pi i/4}$, $e^{5\pi i/4}$, and $e^{7\pi i/4}$ of order 1. Let C be the counterclockwise orientation of the radius R upper-half semicircle centered at 0. Now for $R>1$ we have by the residue theorem that

$$\begin{aligned}\oint_C \frac{z^2}{z^4+1}\,dz &= 2\pi i \operatorname{Res}(f,e^{\pi i/4}) + 2\pi i \operatorname{Res}(f,e^{3\pi i/4}) \\ &= 2\pi i \lim_{z\to\exp(\pi i/4)} \frac{z^2}{(z+e^{\pi i/4})(z-e^{3\pi i/4})(z+e^{3\pi i/4})} \\ &\qquad + 2\pi i \lim_{z\to\exp(3\pi i/4)} \frac{z^2}{(z-e^{\pi i/4})(z+e^{\pi i/4})(z+e^{3\pi i/4})} \\ &= 2\pi i\cdot\frac{i}{(\sqrt{2})(2e^{\pi i/4})(i\sqrt{2})} + 2\pi i\cdot\frac{-i}{(-\sqrt{2})(i\sqrt{2})(2e^{3\pi i/4})} \\ &= \frac{\pi i}{2e^{\pi i/4}} + \frac{\pi i}{2e^{3\pi i/4}} = \frac{\pi i}{2}\left(e^{-\pi i/4}+e^{-3\pi i/4}\right) = \frac{\pi}{\sqrt{2}}\end{aligned}$$

Now let γ_1 be the segment of C along the real axis and let γ_2 be the semicircular part, preserving the orientation of both contours from C. Now note that
$$\int_{\gamma_1}\frac{z^2}{z^4+1}\,dz=\int_{-R}^{R}\frac{z^2}{z^4+1}\,dz.$$
Additionally, note that
$$0\le\left|\int_{\gamma_2}\frac{z^2}{z^4+1}\,dz\right|=\left|iR\int_0^\pi\frac{R^2e^{2i\theta}}{R^4e^{4i\theta}+1}\,e^{i\theta}\,d\theta\right|\le R\int_0^\pi\frac{R^2}{|R^4e^{4i\theta}+1|}\,d\theta.$$
Now note that for $R>1$ we have that
$$R^4-1=\left|R^4-1\right|\le\left|R^4e^{4i\theta}+1\right|.$$
Thus,
$$0\le\left|\int_{\gamma_2}\frac{z^2}{z^4+1}\,dz\right|\le R\int_0^\pi\frac{R^2}{|R^4e^{4i\theta}+1|}\,d\theta\le R\int_0^\pi\frac{R^2}{R^4-1}\,d\theta=\frac{\pi R^3}{R^4-1}.$$
Thus by the squeeze theorem, in the limit $R\to\infty$ we have that the integral over γ_2 vanishes. So,
$$\frac{\pi}{\sqrt{2}}=\lim_{R\to\infty}\oint_C\frac{z^2}{z^4+1}\,dz=\lim_{R\to\infty}\int_{\gamma_1}\frac{z^2}{z^4+1}\,dz+\lim_{R\to\infty}\int_{\gamma_2}\frac{z^2}{z^4+1}\,dz=\int_{-\infty}^{\infty}\frac{z^2}{z^4+1}\,dz.$$

Example 2.4.8 (Spring 2022, Problem 1)**.** Compute the integral
$$\int_{\mathbb{R}}\frac{\cos^2(x)}{x^2+1}\,dx.$$

Solution. Firstly, recall the identity $2\cos^2(z)=1+\cos(2z)$. Thus
$$\int_{\mathbb{R}}\frac{\cos^2(x)}{x^2+1}\,dx=\frac{1}{2}\int_{\mathbb{R}}\frac{1}{x^2+1}\,dx+\frac{1}{2}\int_{\mathbb{R}}\frac{\cos(2x)}{x^2+1}\,dx.$$
Using our solution to Example 2.4.5 we know that
$$\int_{\mathbb{R}}\frac{\cos(2x)}{x^2+1}\,dx=\frac{\pi}{e^2}\implies\int_{\mathbb{R}}\frac{\cos^2(x)}{x^2+1}\,dx=\frac{1}{2}\int_{\mathbb{R}}\frac{1}{x^2+1}\,dx+\frac{\pi}{2e^2}.$$
Let γ be the counterclockwise oriented upper-half circle centered at 0 with radius R. Noting $x^2+1=(x-i)(x+i)$, by the residue theorem we know that for $R>1$ we have
$$\int_\gamma\frac{dz}{z^2+1}=2\pi i\operatorname{Res}(f,i)=2\pi i\lim_{z\to i}\frac{1}{z+i}=\pi.$$

Now let γ' be the arc portion of γ. We parameterize this with $z = Re^{i\theta}$ and $dz = iRe^{i\theta}\,d\theta$ where $0 \le \theta \le \pi$. Thus

$$\int_{\gamma'} \frac{dz}{z^2+1} = \int_0^\pi \frac{iRe^{i\theta}}{R^2 e^{2i\theta}+1}\,d\theta.$$

Now note that

$$\left|\int_{\gamma'} \frac{dz}{z^2+1}\right| = \left|\int_0^\pi \frac{iRe^{i\theta}}{R^2e^{2i\theta}+1}\,d\theta\right| \le \int_0^\pi \left|\frac{iRe^{i\theta}}{R^2e^{2i\theta}+1}\right| d\theta = R\int_0^\pi \frac{1}{|R^2e^{2i\theta}+1|}\,d\theta.$$

But note that for $R > 1$ we have $R^2 - 1 = \left|R^2-1\right| = \left|\left|R^2e^{2i\theta}\right| - 1\right| \le \left|R^2e^{2i\theta}+1\right|$. Thus

$$\left|\int_{\gamma'} \frac{dz}{z^2+1}\right| \le R\int_0^\pi \frac{1}{|R^2e^{2i\theta}+1|}\,d\theta \le R\int_0^\pi \frac{1}{R^2-1}\,d\theta = \frac{\pi R}{R^2-1}.$$

Note that in the limit $R \to \infty$, this integral vanishes. Thus becuase both of the following limits exist, we have

$$\int_{\mathbb{R}} \frac{dz}{z^2+1} = \lim_{R\to\infty}\left(\int_\gamma \frac{dz}{z^2+1} - \int_{\gamma'} \frac{dz}{z^2+1}\right) = \lim_{R\to\infty}\int_\gamma \frac{dz}{z^2+1} - \lim_{R\to\infty}\int_{\gamma'} \frac{dz}{z^2+1} = \pi - 0 = \pi.$$

So we have that

$$\int_{\mathbb{R}} \frac{\cos^2(x)}{x^2+1}\,dx = \frac{1}{2}\int_{\mathbb{R}} \frac{dx}{x^2+1} + \frac{\pi}{2e^2} = \frac{\pi}{2}\left(1+e^{-2}\right).$$

Example 2.4.9 (Fall 2021, Problem 2)**.** For $n \ge 2$ explicitly compute

$$\int_{\mathbb{R}} \frac{x^n}{1+x^{2n}}\,dx.$$

Solution. Note that

$$1 + x^{2n} = (x - e^{i\pi/(2n)})(x - e^{3i\pi/(2n)})\dots(x - e^{(2n-1)\,i\pi/(2n)}).$$

Let γ be the counterclockwise oriented upper-half circle contour centered at 0 with radius R. Note that by the resiude theorem, for $R > 1$ we have

$$\int_\gamma \frac{z^n}{1+z^{2n}}\,dz = 2\pi i \sum_{k=1}^n \operatorname{Res}(f, e^{(2k-1)\,i\pi/(2n)}).$$

Now note that if $z_0 = e^{(2k-1)\,i\pi/(2n)}$ we have that $z_0^{2n} = -1$. Now we evaluate the residue in the general case.

$$\operatorname{Res}(f, z_0) = \lim_{z\to z_0} \frac{(z-z_0)\,z^n}{1+z^{2n}} = \lim_{z\to z_0} \frac{z^{n+1} - z_0 z^n}{1+z^{2n}} = \lim_{z\to z_0} \frac{(n+1)\,z^n - nz_0 z^{n-1}}{2nz^{2n-1}} = \frac{z_0^{n+1}}{2nz_0^{2n}} = -\frac{z_0^{n+1}}{2n}.$$

Thus we have that

$$\int_\gamma \frac{z^n}{1+z^{2n}}\,dz = -\frac{\pi i}{n}\sum_{k=1}^n \exp\left(\frac{(2k-1)(n+1)\,\pi i}{2n}\right) = -\frac{\pi}{n}\sum_{k=1}^n \exp\bigl(\pi i k\,(1+n^{-1})\bigr).$$

Now let γ' be the arc portion of the contour γ. Now we parametrize $z = Re^{i\theta}$ and $dz = iRe^{i\theta}\,d\theta$ and note

$$\int_{\gamma'} \frac{z^n}{1+z^{2n}}\,dz = \int_0^\pi \frac{R^n e^{in\theta}}{1+R^{2n}e^{2in\theta}}\,iRe^{i\theta}d\theta.$$

Now note that

$$\left|\int_{\gamma'} \frac{z^n}{1+z^{2n}}\,dz\right| = \left|\int_0^\pi \frac{R^n e^{in\theta}}{1+R^{2n}e^{2in\theta}}\,iRe^{i\theta}d\theta\right| \le R^{n+1}\int_0^\pi \frac{d\theta}{|1+R^{2n}e^{2in\theta}|}.$$

But now note that for $R > 1$ we have that

$$R^{2n} - 1 = \left|R^{2n}-1\right| = \left|\left|R^{2n}e^{2in\theta}\right| - 1\right| \le \left|1 + R^{2n}e^{2in\theta}\right|.$$

Thus

$$\left|\int_{\gamma'} \frac{z^n}{1+z^{2n}}\,dz\right| \le R^{n+1}\int_0^\pi \frac{d\theta}{|1+R^{2n}e^{2in\theta}|} \le R^{n+1}\int_0^\pi \frac{d\theta}{R^{2n}-1} = \frac{R^{n+1}}{R^{2n}-1}.$$

Note that when $n \geq 2$ the right hand side goes to 0 in the limit $R \to \infty$. Thus because both limits exist, we have

$$\begin{aligned}
\int_{\mathbb{R}} \frac{z^n}{1+z^{2n}}\,dz &= \lim_{R\to\infty}\left(\int_\gamma \frac{z^n}{1+z^{2n}}\,dz - \int_{\gamma'} \frac{z^n}{1+z^{2n}}\,dz\right) \\
&= \lim_{R\to\infty}\int_\gamma \frac{z^n}{1+z^{2n}}\,dz - \lim_{R\to\infty}\int_{\gamma'} \frac{z^n}{1+z^{2n}}\,dz = -\frac{\pi}{n}\sum_{k=1}^{n}\exp\bigl(\pi i k(1+n^{-1})\bigr) - 0 = -\frac{\pi}{n}\sum_{k=1}^{n}\exp\bigl(\pi i k(1+n^{-1})\bigr).
\end{aligned}$$

(Unsure where to go from here)

Example 2.4.10 (Spring 2021, Problem 3)**.** Evaluate the integral

$$\int_{\mathbb{R}} \frac{\cos x}{x^4 - (\pi/2)^4}\,dx.$$

Solution. Let γ be the semicircular contour centered at 0 with radius $R > \pi/2$ with semicircular indents along the real axis centered at $\pm\pi/2$ with radius $1/R$. Let γ^- be the semicircular indent centered at $-\pi/2$ and let γ^+ be the semicircular indent centered at $\pi/2$. Similarly, let γ' be the main semicircular arc. Now note that

$$\int_\gamma f = \int_{-\pi/2+1/R}^{\pi/2-1/R} f + \int_{\gamma^+} f + \int_{\pi/2+1/R}^{R} f + \int_{\gamma'} f + \int_{-R}^{-\pi/2-1/R} f + \int_{\gamma^-} f \qquad \text{where} \qquad f(z) = \frac{e^{iz}}{z^4-(\pi/2)^4}.$$

Now by the residue theorem we have that

$$\int_\gamma f = 2\pi i\,\mathrm{Res}(f, i\pi/2) = 2\pi i \lim_{z\to i\pi/2} \frac{e^{iz}}{(z+i\pi/2)(z^2-(\pi/2)^2)} = 4\pi i\left(\frac{\exp(-\pi/2)}{-i\pi^3}\right) = -\frac{4}{\pi^2}\exp(-\pi/2).$$

Now note that

$$\left|\int_{\gamma'} f\right| = \left|\int_0^\pi \frac{\exp\bigl(iRe^{i\theta}\bigr)}{R^4e^{4i\theta}-(\pi/2)^4}\, iRe^{i\theta}\,d\theta\right| \leq R\int_0^\pi \frac{\exp(-R\sin\theta)}{R^4-(\pi/2)^4}\,d\theta.$$

Since $0 \leq \theta \leq \pi$ we have that $\exp(-R\sin\theta) \leq 1$. So,

$$\left|\int_{\gamma'} f\right| \leq R\int_0^\pi \frac{\exp(-R\sin\theta)}{R^4-(\pi/2)^4}\,d\theta \leq R\int_0^\pi \frac{d\theta}{R^4-(\pi/2)^4} = \frac{\pi R}{R^4-(\pi/2)^4}.$$

So we have that $\int_{\gamma'} f \to 0$ as $R \to \infty$. Similarly we have that

$$\int_{\gamma^-} f = \int_\pi^0 \frac{\exp\bigl(-i\pi/2 + ie^{i\theta}/R\bigr)}{(-\pi/2+e^{i\theta}/R)^4-(\pi/2)^4}\cdot\frac{ie^{i\theta}}{R}\,d\theta = \int_{-\pi/2-1/R}^{-\pi/2+1/R} \frac{\exp(iu)}{u^4-(\pi/2)^4}\,du.$$

But now note that

$$\int_{\gamma^-} f = \int_{-\pi/2-1/R}^{-\pi/2+1/R} \frac{\exp(iu)}{u^4-(\pi/2)^4}\,du = \int_{-\pi/2-1/R}^{-\pi/2+1/R} \frac{\cos u\,du}{u^4-(\pi/2)^4} + i\int_{-\pi/2-1/R}^{-\pi/2+1/R} \frac{\sin u\,du}{u^4-(\pi/2)^4}.$$

Similarly,

$$\int_{\gamma^+} f = \int_{\pi/2-1/R}^{\pi/2+1/R} \frac{\exp(iu)}{u^4-(\pi/2)^4}\,du = \int_{\pi/2-1/R}^{\pi/2+1/R} \frac{\cos u\,du}{u^4-(\pi/2)^4} + i\int_{\pi/2-1/R}^{\pi/2+1/R} \frac{\sin u\,du}{u^4-(\pi/2)^4}.$$

But note that

$$\int_{\pi/2-1/R}^{\pi/2+1/R} \frac{\sin u\,du}{u^4-(\pi/2)^4} = -\int_{-\pi/2+1/R}^{-\pi/2-1/R} \frac{\sin(-u)\,du}{(-u)^4-(\pi/2)^4} = -\int_{-\pi/2-1/R}^{-\pi/2+1/R} \frac{\sin u\,du}{u^4-(\pi/2)^4}.$$

Thus,

$$\int_{\gamma^-} f + \int_{\gamma^+} f = \int_{-\pi/2-1/R}^{\pi/2+1/R} \frac{\cos u\,du}{u^4-(\pi/2)^4} + \int_{\pi/2-1/R}^{\pi/2+1/R} \frac{\cos u\,du}{u^4-(\pi/2)^4} = \int_{-\pi/2-1/R}^{-\pi/2+1/R} \mathfrak{Re}\,f + \int_{\pi/2-1/R}^{\pi/2+1/R} \mathfrak{Re}\,f.$$

Note that this is purely a real valued integral. Now taking the real part of our integral summation yields

$$-\frac{4}{\pi^2}\exp(-\pi/2) = \int_\gamma \mathfrak{Re}\, f = \int_{\gamma'} \mathfrak{Re}\, f + \mathfrak{Re}\left(\int_{\gamma^-} f + \int_{\gamma^+} f\right) + \int_{-R}^{-\pi/2-1/R} \mathfrak{Re}\, f + \int_{-\pi/2+1/R}^{\pi/2-1/R} \mathfrak{Re}\, f + \int_{\pi/2+1/R}^{R} \mathfrak{Re}\, f.$$

Thus, by substituting the previous expression and noting that it is purely real, we have

$$\begin{aligned} -\frac{4}{\pi^2}\exp(-\pi/2) &= \int_{\gamma'} \mathfrak{Re}\, f + \int_{-R}^{-\pi/2-1/R} \mathfrak{Re}\, f + \int_{-\pi/2-1/R}^{-\pi/2+1/R} \mathfrak{Re}\, f + \int_{-\pi/2+1/R}^{\pi/2-1/R} \mathfrak{Re}\, f + \int_{\pi/2-1/R}^{\pi/2+1/R} \mathfrak{Re}\, f + \int_{\pi/2+1/R}^{R} \mathfrak{Re}\, f \\ &= \int_{\gamma'} \mathfrak{Re}\, f + \int_{-R}^{R} \mathfrak{Re}\, f. \end{aligned}$$

Note that in the limit $R \to \infty$ the term over γ' vanishes, thus

$$-\frac{4}{\pi}\exp(-\pi/2) = \int_{\mathbb{R}} \mathfrak{Re}\, f = \int_{\mathbb{R}} \frac{\cos x\, dz}{x^4 - (\pi/2)^4}.$$

Example 2.4.11 (Fall 2020, Problem 1)**.** Evaluate the integral

$$\int_{\mathbb{R}} \frac{\cos x}{1 + x + x^2}\, dx.$$

Solution. Note that

$$1 + x + x^2 = \left(x - (-1 + i\sqrt{3})/2\right)\left(x - (-1 - i\sqrt{3})/2\right).$$

Thus by taking γ as the semicircular contour centered at $-1/2$ with radius $R > \sqrt{3}/2$, we have by residue theorem

$$\int_\gamma \frac{e^{iz}}{1 + z + z^2}\, dz = 2\pi i \lim_{z \to (-1+i\sqrt{3})/2} \frac{e^{iz}}{(z - (-1 - i\sqrt{3})/2)} = \frac{2\pi \exp(-i/2)\, \exp\left(-\sqrt{3}/2\right)}{\sqrt{3}}.$$

Note that

$$\int_\gamma \frac{\cos z}{1 + z + z^2}\, dz = \mathfrak{Re}\left(\int_\gamma \frac{e^{iz}}{1 + z + z^2}\, dz\right) = \mathfrak{Re}\left(\frac{2\pi \exp(-i/2)\exp\left(-\sqrt{3}/2\right)}{\sqrt{3}}\right) = \frac{2\pi \cos(1/2)\exp\left(-\sqrt{3}/2\right)}{\sqrt{3}}.$$

Now if we let $z = Re^{i\theta}$ and $dz = iRe^{i\theta} d\theta$ to parametrize the arc (which we denote γ'), we have that

$$\left|\int_{\gamma'} \frac{e^{iz}}{1 + z + z^2}\, dz\right| = \left| iR \int_0^\pi \frac{\exp\left(iRe^{i\theta}\right)}{1 + Re^{i\theta} + R^2 e^{2i\theta}}\, e^{i\theta}\, d\theta\right| \le R \int_0^\pi \frac{\exp(-R\sin\theta)}{|1 + Re^{i\theta} + R^2 e^{2i\theta}|}\, d\theta.$$

Now note that for sufficently large R we have

$$R^2 - R - 1 = R^2 - \left(\left|Re^{i\theta}\right| + 1\right) \le R^2 - \left|Re^{i\theta} + 1\right| = \left|R^2 - \left|Re^{i\theta} + 1\right|\right| = \left|\left|R^2 e^{2i\theta}\right| - \left|Re^{i\theta} + 1\right|\right| \le \left|1 + Re^{i\theta} + R^2 e^{2i\theta}\right|.$$

Thus because $0 \le \theta \le \pi$ we have that $\exp(-R\sin\theta) \le 1$ and thus

$$\left|\int_{\gamma'} \frac{e^{iz}}{1 + z + z^2}\, dz\right| \le R \int_0^\pi \frac{\exp(-R\sin\theta)}{|1 + Re^{i\theta} + R^2 e^{2i\theta}|}\, d\theta \le R \int_0^\pi \frac{d\theta}{R^2 - R - 1} = \frac{\pi R}{R^2 - R - 1}.$$

Now note that in the limit $R \to \infty$ we have that this upper bound vanishes. So by the squeeze theorem, we know that the integral over γ' necessarily vanishes also. Since the integral with $\cos z$ over γ' is merely the real part of the above integral, we know this integral must also vanish. Thus because both limits exist and are finite, we have

$$\begin{aligned} \int_{\mathbb{R}} \frac{\cos z}{1 + z + z^2}\, dz &= \lim_{R\to\infty}\left(\int_\gamma \frac{\cos z}{1 + z + z^2}\, dz - \int_{\gamma'} \frac{\cos z}{1 + z + z^2}\, dz\right) \\ &= \lim_{R\to\infty} \int_\gamma \frac{\cos z}{1 + z + z^2}\, dz - \lim_{R\to\infty} \int_{\gamma'} \frac{\cos z}{1 + z + z^2}\, dz = \frac{2\pi \cos(1/2)\exp\left(-\sqrt{3}/2\right)}{\sqrt{3}}. \end{aligned}$$

2.5 Argument Principle

Theorem 2.5.1. *If f is a meromorphic function inside some closed contour C, and f has no zeros or poles on C itself, then we have*

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = Z - P$$

where Z and P are the number of zeros and poles respectively of f inside C.

Remark. *For any contour γ and meromorphic function f with no zeros or poles on γ, we loosely have that*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{\Delta_\gamma \arg(f(z))}{2\pi}.$$

(1) Example Problems: Argument Principle

Example 2.5.2 (Fall 2022, Problem 4)**.** Let D be a domain in $\mathbb{C}$ and let f be a holomorphic function in D. Suppose that $\mathrm{Re}(f) > 0$. Prove that for any closed C^1-piecewise smooth curve C,

$$\oint_C \frac{f'}{f} dz = 0$$

Additionally, use the argument principle to prove that for any $\lambda > 0, p(z) = z^4 + i\lambda z^3 + 1 = 0$ has exactly one solution in the first quadrant.

Solution. Suppose that there exists $z \in \mathbb{D}$ such that $f(z) = 0$. Then $\mathfrak{Re}(f(z)) = 0$ as well which contradicts the fact that $\mathfrak{Re}(f) > 0$. Thus f has no zeros on D. Additionally, f has no poles on D, as it is holomorphic. Thus by the argument principle we know that

$$\frac{1}{2\pi i}\oint_C \frac{p'}{p}\,dz = Z - P = 0 - 0 = 0. \implies \oint_C \frac{p'}{p}\,dz = 0.$$

Noting that $P(z)$ is holomorphic on the plane, we have by the argument principle that

$$\frac{1}{2\pi i}\oint_C \frac{p'}{p}\,dz = Z - P = Z$$

for any closed contour C. Now let C be the clockwise oriented quarter circle of radius R in the first quadrant. Let γ_1 be the path along the real axis between $0 \to R$, let γ_2 be the path along the quarter circle of radius R centered at 0 between $R \to iR$, and let γ_3 be the path along the imaginary axis between $iR \to 0$. Note

$$Z = \frac{1}{2\pi i}\oint_C \frac{p'}{p}\,dz = \frac{1}{2\pi i}\left(\int_{\gamma_1} \frac{p'}{p}\,dz + \int_{\gamma_2} \frac{p'}{p}\,dz + \int_{\gamma_3} \frac{p'}{p}\,dz\right) = \frac{\Delta_{\gamma_1}\arg(p(z)) + \Delta_{\gamma_2}\arg(p(z)) + \Delta_{\gamma_3}\arg(p(z))}{2\pi}$$

Now note that for $x \in \mathbb{R}$ we have

$$p(ix) = (ix)^4 + i\lambda(ix)^3 + 1 = x^4 + \lambda x^3 + 1 \in \mathbb{R} \implies \Delta_{\gamma_3}\arg(p(z)) = 0.$$

It also follows from the above that

$$\Delta_{\gamma_1}\arg(p(z)) = \arg(p(R)) - \arg(p(0)) = \arg(p(R)) = \arg(R^4) + \arg(1 + i\lambda/R + 1/R^4) = \arg(1 + i\lambda/R + 1/R^4).$$

From this it is clear that in the limit $R \to \infty$ we have that $\Delta_{\gamma_1}\arg(p(z)) = 0$. Now for $0 \le \theta \le \pi/2$ note that

$$\arg(p(Re^{i\theta})) = \arg(R^4e^{4i\theta}) + \arg(1 + i\lambda/(Re^{i\theta}) + 1/(R^4e^{4i\theta})) = 4\theta + \arg(1 + i\lambda/(Re^{i\theta}) + 1/(R^4e^{4i\theta})).$$

From this it is clear that in the limit $R \to \infty$ we have that $\arg(p(Re^{i\theta})) = 4\theta$. Thus in the limit $R \to \infty$ we have that $\Delta_{\gamma_2}\arg(p(z)) = 2\pi$. Thus we have

$$\lim_{R\to\infty} Z = \lim_{R\to\infty} \frac{\Delta_{\gamma_1}\arg(p(z)) + \Delta_{\gamma_2}\arg(p(z)) + \Delta_{\gamma_1}\arg(p(z))}{2\pi} = \frac{0 + 2\pi + 0}{2\pi} = 1.$$

2.6 Biholomorphic Mappings

Lemma 2.6.1. *The Cayley transform maps the upper-half plane to the unit disk.*

$$f : \mathcal{H} \to \mathbb{D} \qquad z \mapsto \frac{z-i}{z+i}$$

Lemma 2.6.2. *This Mobius transformation maps the disk to itself. For $a \in \mathbb{D}$ and $0 \le \theta < 2\pi$ we have*

$$f : \mathbb{D} \to \mathbb{D} \qquad z \mapsto e^{i\theta}\left(\frac{z-a}{1-\overline{a}z}\right).$$

Note that $f(a) = 0$.

Lemma 2.6.3. *This transformation take the strip $\mathbb{S} = \{z \in \mathbb{C} : |\mathfrak{Im}(z)| < 1\}$ to the right half plane $-i\mathcal{H}$.*

$$f : \mathbb{S} \to -i\mathcal{H} \qquad z \mapsto \exp(\pi z/2).$$

Lemma 2.6.4. *Trigonometric functions can take the half-strip $\mathbb{S}^+ = \{z \in \mathbb{C} : |\mathfrak{Re}(z)| < 1, \, \mathfrak{Im}(z) > 0\}$ to $\mathcal{H}$:*

$$f : \mathbb{S}^+ \to \mathcal{H} \qquad z \mapsto \sin(\pi z/2).$$

Theorem 2.6.5 (Riemann Mapping)**.** *If U is a non-empty, simply connected subset of $\mathbb{C}$ that is not itself $\mathbb{C}$, then there exists a biholomorphic mapping $f : U \to \mathbb{D}$.*

Theorem 2.6.6 (Caratheodory's Theorem)**.** *If we have a conformal map $f : \mathbb{D} \to U$ where U is simply connected in $\mathbb{C} \cup \{\infty\}$ and ∂U is a Jordan curve in $\mathbb{C} \cup \{\infty\}$, then there exists a continuous extension of f to $g : \overline{\mathbb{D}} \to \overline{U}$ which is also one-to-one.*

(3) Example Problems: Biholomorphic Mappings

Example 2.6.7 (Spring 2023, Problem 3)**.** Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{A} = \{z \in \mathbb{C} : 0 < \arg z < 2\pi/5\}$. Find an explicit biholomoprhism $f : \mathbb{D} \to \mathbb{A}$.

Solution. Note that we have the biholomorphism

$$f : \mathbb{A} \to \mathcal{H} \qquad z \mapsto z^{5/2} \qquad \text{with inverse} \qquad f^{-1} : \mathcal{H} \to \mathbb{A} \qquad z \mapsto z^{2/5}.$$

Likewise, recall the biholomorphism given by the Cayley transform

$$g : \mathcal{H} \to \mathbb{D} \qquad z \mapsto \frac{z-i}{z+i} \qquad \text{with inverse} \qquad g^{-1} : \mathbb{D} \to \mathcal{H} \qquad z \mapsto i\left(\frac{1+z}{1-z}\right).$$

Composing biholomorphisms we have

$$g \circ f : \mathbb{A} \to \mathbb{D} \qquad z \mapsto \frac{z^{5/2}-i}{z^{5/2}+i} \qquad \text{with inverse} \qquad f^{-1} \circ g^{-1} : \mathbb{D} \to \mathbb{A} \qquad z \mapsto \left(i\left(\frac{1+z}{1-z}\right)\right)^{2/5}.$$

Example 2.6.8 (Fall 2020, Problem 3)**.** Construct a conformal map from $\mathbb{S}^- := \{z \in \mathbb{C} : \mathfrak{Re}(z) < 0, 0 < \mathfrak{Im}(z) < 1\}$ to the upper-half plane such that it has a continuous extension to the closure of $\mathbb{S}^-$ considered as a map to the extended complex plane, and fixes 0. You may construct the map as a composition of elementary conformal maps.

Solution. Let $\mathbb{S}^+ = \{z \in \mathbb{C} : |\mathfrak{Re}(z)| < 1, \, \mathfrak{Im}(z) > 0\}$, then we have the conformal map

$$f : \mathbb{S}^- \to \mathbb{S}^+ \qquad z \mapsto -1 - 2zi.$$

We also have the elementary conformal map

$$g : \mathbb{S}^+ \to \mathcal{H} \qquad z \mapsto \sin(\pi z/2).$$

We also have the extremely esoteric conformal map

$$h : \mathcal{H} \to \mathcal{H} \qquad z \mapsto z + 1.$$

Note the composition

$$h \circ g \circ f : \mathbb{S}^- \to \mathcal{H} \qquad z \mapsto 1 + \sin\left(-\frac{\pi(1+2zi)}{2}\right) \qquad \text{with} \qquad (h \circ g \circ f)(0) = 0.$$

Noting that both $\mathbb{S}^-$ and $\mathcal{H}$ are simply connected in $\mathbb{C} \cup \{\infty\}$ and that $\partial\mathbb{S}^-$ and $\partial\mathcal{H}$ are Jordan curves in $\mathbb{C} \cup \{\infty\}$ allows us to conclude via Caratheodory's theorem that $h \circ g \circ f$ admits a continuous extension from $\overline{\mathbb{S}^-}$ to $\overline{\mathcal{H}}$.

Example 2.6.9 (Fall 2023, Problem 3)**.** Does there exist a holomorphic surjection from $\mathbb{D}$ to $\mathbb{C}$.

Solution. Yes, consider the inverse cayley transform

$$f : \mathbb{D} \to \mathcal{H} \qquad \text{such that} \qquad z \mapsto i\left(\frac{1+z}{1-z}\right).$$

Now for $\varepsilon > 0$ we have

$$g : \mathcal{H} \to \mathcal{H} - i\varepsilon \qquad \text{such that} \qquad z \mapsto z - i\varepsilon.$$

And finally, we square $\mathbb{H} - i\varepsilon$ to map to the whole complex plane.

$$h : \mathcal{H} - i\varepsilon \to \mathbb{C} \qquad \text{such that} \qquad z \mapsto z^2.$$

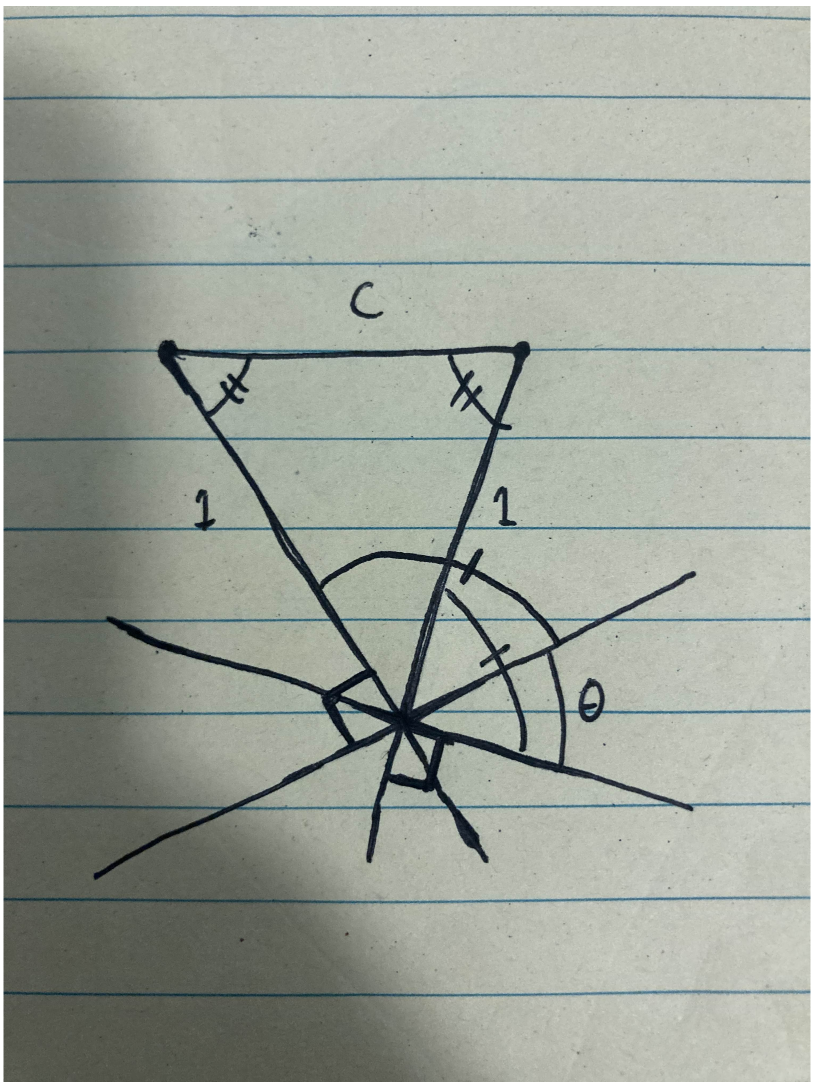
Our holomorphic surjection is simply the composition of these maps where $\phi = h \circ g \circ f$.

Example 2.6.10 (Spring 2024, Problem 4)**.** Let $c > 0$ and

$$D = \{|z| > 1, |z - c| < 1\}, \qquad F(z) = \frac{z - z_1}{z - z_2}$$

where $z_1, z_2 \in \mathbb{C}$ are the intersection points of the circles $|z| > 1$ and $|z - c| = 1$, with $\mathfrak{Im}\, z_1 < 0$ and $\mathfrak{Im}\, z_2 > 0$. Find the value of c such that $F(D)$ is bounded by two rays with angle equal to $\pi/3$. Then find $F(D)$.

Solution. Note that $F(z_1) = 0$ and $F(z_2) = \infty$. Now note that F is a conformal map, and thus preserves angles (and similarly for F^{-1}); so we have the geometry



By simply examining the diagram, we note that the bottom angle of the triangle is also $\theta = \pi/3$. Thus the double hatched angles are also $\pi/3$, and we have an equilateral triangle between the centers of the circles and z_1. Thus, $c = 1$. Therefore $z_1 = (1 - i\sqrt{3})/2$ and $z_2 = (1 + i\sqrt{3})/2$. Now note that when $z' = 1$ we have

$$F(z') = \frac{z' - z_1}{z' - z_2} = \frac{1 - (1 - i\sqrt{3})/2}{1 - (1 + i\sqrt{3})/2} = \frac{(1 + i\sqrt{3})/2}{(1 - \sqrt{3})/2} = \frac{e^{\pi i/3}}{e^{-\pi i/3}} = e^{2\pi i/3} \qquad \text{and} \qquad |z'| = 1.$$

Thus because $F(z_1) = 0$, $F(z') = e^{2\pi i/3}$, and $F(z_2) = \infty$, this arc of the circle centered at 0 maps to the ray $\arg z = 2\pi/3$. Now note that when $z'' = (3 + i\sqrt{3})/2$ we have

$$F(z'') = \frac{z'' - z_1}{z'' - z_2} = \frac{(3 + i\sqrt{3})/2 - (1 - i\sqrt{3})/2}{(3 + i\sqrt{3})/2 - (1 + i\sqrt{3})/2} = 1 + i\sqrt{3} = 2e^{\pi i/3} \qquad \text{and} \qquad |z'' - c| = 1.$$

Thus because $F(z_1) = 0$, $F(z'') = 2e^{\pi i/3}$, and $F(z_2) = \infty$, this arc of the circle centered at c maps to the ray $\arg z = \pi/3$. Thus we have that

$$f(D) = \{z \in \mathbb{C} : \pi/3 < \arg z < 2\pi/3\}.$$

2.7 Schwarz Lemma

Lemma 2.7.1. *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map such that $f(0) = 0$. Then it follows that $|f(z)| \leq |z|$ for $z \in \mathbb{D}$ and $|f'(0)| \leq 1$.*

Corollary 2.7.2. *Furthermore, under the same conditions, if $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$, then $f(z) = az$ for some $z \in \partial\mathbb{D}$.*

(2) Example Problems: Schwarz Lemma

Example 2.7.3 (Spring 2022, Problem 4)**.** Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with two fixed points. Show that f is precisely the identity map.

Solution. For the sake of notation let $f(a) = a$ and $f(b) = b$ for some distinct $a, b \in \mathbb{D}$. Now we define the biholomorphic map

$$g : \mathbb{D} \to \mathbb{D} \qquad \text{such that} \qquad z \mapsto \frac{z-a}{1-\overline{a}z}.$$

Note that the inverse map is precisely

$$g^{-1} : \mathbb{D} \to \mathbb{D} \qquad \text{such that} \qquad z \mapsto \frac{z+a}{1+\overline{a}z}.$$

Now we define the holomorphic map $h = g \circ f \circ g^{-1} : \mathbb{D} \to \mathbb{D}$. Note $h(0) = g(f(g^{-1}(0))) = g(f(a)) = g(a) = 0$ and

$$h\left(\frac{b-a}{1-\overline{a}b}\right) = g\left(f\left(g^{-1}\left(\frac{b-a}{1-\overline{a}b}\right)\right)\right) = g(f(b)) = g(b) = \frac{b-a}{1-\overline{a}b}.$$

Thus by the corollary to Schwarz lemma, we know that $h(z) = cz$ for some $c \in \partial\mathbb{D}$. But since

$$h\left(\frac{b-a}{1-\overline{a}b}\right) = \frac{b-a}{1-\overline{a}b} \qquad \text{and} \qquad a \neq b,$$

we also know that $h(z) = z$. Thus,

$$z = h(z) = g(f(g^{-1}(z))) \implies g(z) = g(f(g^{-1}(g(z)))) = g(f(z)) \implies z = g^{-1}(g(z)) = g^{-1}(g(f(z))) = f(z).$$

Example 2.7.4 (Spring 2021, Problem 2)**.** Let $f : \mathcal{H} \to \mathbb{C}$ be a holomorphic function such that $|f(z)| \leq 1$ and $f(i) = 0$. Prove that for $z \in \mathcal{H}$ that

$$|f(z)| \leq \left|\frac{z-i}{z+i}\right|.$$

Solution. Let us define the Cayley transform

$$g : \mathcal{H} \to \mathbb{D} \qquad \text{such that} \qquad z \mapsto \frac{z-i}{z+i}.$$

Note that $g(i) = 0$, so $g^{-1}(0) = i$. Now let $h = f \circ g^{-1} : \mathbb{D} \to \mathbb{C}$. Note that as before $|h(z)| \leq 1$ and also $h(0) = 0$. By Schwarz lemma we have that $|h(z)| \leq |z|$ for $z \in \mathbb{D}$. By extension we have that for $z \in \mathcal{H}$ it follows that

$$|f(z)| = \left|f(g^{-1}(g(z))\right| = |h(g(z))| \leq |g(z)| = \left|\frac{z-i}{z+i}\right|.$$

2.8 Maximum Modulus Principle

Theorem 2.8.1. *Suppose that $\Omega \subset \mathbb{C}$ is a non-empty open connected subset and that f is a non-constant holomorphic function on Ω. It then follows that f cannot attain a maximum in Ω.*

Corollary 2.8.2. *Suppose that $\Omega \subset \mathbb{C}$ is a non-empty open connected subset with compact closure $\overline{\Omega}$. If f is holomorphic on Ω and continuous on $\overline{\Omega}$, then*

$$\sup_{z\in\Omega} |f(z)| \leq \sup_{z\in\partial\Omega} |f(z)|.$$

(4) Example Problems: Maximum Modulus Principle

Example 2.8.3 (Spring 2023, Problem 4)**.** Let $\mathbb{S} := \{z = x + iy : -1 \leqslant x \leqslant 1\}$ and let $f : \mathbb{S} \to \mathbb{C}$ be a bounded continuous function that is holomorphic on the interior of the strip $\mathbb{S}$. For $-1 \leqslant x \leqslant 1$ let $M(x) := \sup_{y\in\mathbb{R}} |f(x+iy)|$.

- Suppose $M(1), M(-1) \leqslant 1$. Prove that $|f(z)| \leqslant 1$ for any $z \in \mathbb{S}$.
- Suppose $M(1), M(-1)$ are arbitrary. Prove that $M(0)^2 \leqslant M(-1) \cdot M(1)$ by deducing it from part 1 of this problem.

Solution. Let $\varepsilon > 0$ and let

$$F_\varepsilon(z) = f(z) \cdot e^{\varepsilon z^2}.$$

Since $f(z)$ is bounded there exists M such that $|f(z)| \leq M$ for all $z \in \mathbb{S}$. Now note that

$$|F_\varepsilon(z)| = |f(z)| \cdot \exp\big(\mathfrak{Re}(\varepsilon z^2)\big) = |f(z)| \cdot \exp\big(\varepsilon\,(x^2 - y^2)\big) \leq M \cdot \exp\big(\varepsilon\,(1 - y^2)\big).$$

Thus it follows that as $|z| \to \infty$ we have that $|F_\varepsilon(z)| \to 0$. Let $\mathbb{S}_y = \{z \in \mathbb{S} : |\mathfrak{Im}(z)| \leq y\}$ and note that because of this limiting condition, it follows that for every $\varepsilon > 0$ there exists y sufficiently large such that $\sup_{z\in\partial\mathbb{S}_y} |F_\varepsilon(z)| \leq 1$ and $|F_\varepsilon(z)| < 1$ for all $z \in \mathbb{S} \setminus \mathbb{S}_y$. Thus by the maximum modulus principle we have that

$$\sup_{z\in\mathbb{S}_y} |F_\varepsilon(z)| \leq \sup_{z\in\partial\mathbb{S}_y} |F_\varepsilon(z)| \leq 1.$$

Since $|F_\varepsilon(z)| < 1$ for all $z \in \mathbb{S} \setminus \mathbb{S}_y$ also, we have that $\sup_{z\in\mathbb{S}} |F_\varepsilon(z)| \leq 1$. Now when $\varepsilon \to 0$ we have $\sup_{z\in\mathbb{S}} |f(z)| \leq 1$. Thus for all $z \in \mathbb{S}$ we have that $|f(z)| \leq 1$.

SECOND PART???

Example 2.8.4 (Fall 2023, Problem 4)**.** Let $z_1, z_2, \ldots, z_n$ be points on the unit circle in the complex plane. Prove that there exists a point z on the unit circle such that

$$\prod_{k=1}^{n} |z - z_k| = 1.$$

Solution. Let us define the holomorphic function $f(z) = \prod_{k=1}^{n} z - z_k$ on $\mathbb{D}$ and note that it is continuous on $\overline{\mathbb{D}}$. Note that $f(0) = 1$. So, by the maximum modulus principle, we have that

$$1 \leq \sup_{z\in\mathbb{D}} |f(z)| \leq \sup_{z\in\partial\mathbb{D}} |f(z)|.$$

So there exists $z_0 \in \partial\mathbb{D}$ such that $1 \leq |f(z_0)|$. Now note that $|f(z_1)| = 0$. Since f is continuous when parametrized over the unit circle, by the IVT we have that there must exist a point $z \in \partial\mathbb{D}$ such that $|f(z)| = 1$ as desired.

Example 2.8.5 (Fall 2021, Problem 3)**.** Let $D_0 = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and $f : D_0 \to \mathbb{C}$ be holomorphic on D_0 and satisfy $|f(z)| \leq \log(1/|z|)$ for all $z \in D_0$. Prove that $f \equiv 0$ on D

Solution. Let $g(z) = zf(z)$ and note that $|g(z)| \leq |z| \log(1/|z|)$. Note

$$\lim_{x\to 0} x \log(1/x) = \lim_{x\to 0} \frac{\log(1/x)}{1/x} = \lim_{x\to 0} \frac{-1/x}{-1/x^2} = \lim_{x\to 0} x = 0.$$

Thus we have that $|g(z)| \to 0$ as $|z| \to 0$. Thus by the maximum modulus principle $g(z)$ must attain its maximum on ∂D_0, but is necessarily 0 on ∂D_0. So we have that $g(z) \equiv 0$ on D_0 as thus $f(z) \equiv 0$ on D_0.

Example 2.8.6 (August 2020, Problem 5)**.** Let f be holomorphic on a neighborhood of the closed unit disc centered at the origin. Assume that $|f(z)| = 1$ if $|z| = 1$, and is not a constant on the disc. Prove that there exist a positive integer k, points $\alpha_1, \ldots, \alpha_n$ in the open unit disc, positive integers $m_1, \ldots, m_n$, and a complex number β with $|\beta| = 1$ such that

$$f(z) = \beta \prod_{k=1}^{n} \left(\frac{z - \alpha_k}{1 - \overline{\alpha}_k z} \right)^{m_k} \qquad \text{for all } z \text{ in the unit disc.}$$

Solution. We know that f must have finitely many zeros on the disk by the identity theorem. So let $a_1, \ldots, a_n$ be the zeros of f with multiplicities $m_1, \ldots, m_n$. First note that when $|z| = 1$ we have

$$\begin{aligned} |B(z; a_k, m_k)|^2 = B(z; a_k, m_k)\, \overline{B(z; a_k, m_k)} = \left(\frac{z - a_k}{1 - \overline{a_k} z} \right)^{m_k} \left(\frac{\overline{z} - \overline{a_k}}{1 - a_k \overline{z}} \right)^{m_k} &= \left(\frac{|z|^2 - \overline{a_k} z - a_k \overline{z} + |a_k|^2}{1 - a_k \overline{z} - \overline{a_k} z + |a_k|^2 \cdot |z|^2} \right)^{m_k} \\ &= \left(\frac{1 - a_k \overline{z} - \overline{a_k} z + |a_k|^2}{1 - a_k \overline{z} - \overline{a_k} z + |a_k|^2} \right)^{m_k} = 1. \end{aligned}$$

Thus $|B(z; a_k, m_k)|^2 = 1$ and so $|B(z; a_k, m_k)| = 1$ when $|z| = 1$. Now note that by extension

$$\left| \frac{f(z)}{\prod_{k=1}^{n} B(z; a_k, m_k)} \right| = 1 \qquad \text{when} \qquad |z| = 1.$$

Additionally, this function must never vanish on $\mathbb{D}$ by construction. Thus by the minimum modulus principle, this function must be a constant of absolute value 1, denote this constant β. Thus

$$\frac{f(z)}{\prod_{k=1}^{n} B(z; a_k, m_k)} = \beta \implies f(z) = \beta \prod_{k=1}^{n} B(z; a_k, m_k) = \beta \prod_{k=1}^{n} \left(\frac{z - a_k}{1 - \overline{a_k} z} \right)^{m_k}.$$

2.9 Mean Value Theorem

Theorem 2.9.1. *If u is a harmonic function on U and $B(a,r) \subset U$, then we have that*

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})\, d\theta.$$

(1) Example Problems: Mean Value Theorem

Example 2.9.2 (Spring 2022, Problem 3)**.** Suppose $f(z)$ is an entire function such that $\iint_{\mathbb{C}} |f'(z)|^2\, dx\, dy < \infty$. Show that f is constant.

Solution. First we will show a corollary of the MVT:

$$u(a) = \frac{1}{\pi r^2} \iint_{B(a,r)} u(x,y)\, dx\, dy.$$

Let $a = x' + iy'$ and note using Jacobians we have

$$\frac{1}{\pi r^2} \iint_{B(a,r)} u(x,y)\, dx\, dy = \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} u(x' + r'\cos\theta, y' + r'\sin\theta) \cdot \det \begin{pmatrix} \cos\theta & -r'\sin\theta \\ \sin\theta & r'\cos\theta \end{pmatrix} d\theta\, dr'.$$

Simplifying and rearranging we have

$$\frac{1}{\pi r^2} \iint_{B(a,r)} u(x,y)\, dx\, dy = \frac{1}{\pi r^2} \int_0^r r' \int_0^{2\pi} u(x' + r'\cos\theta, y' + r'\sin\theta)\, d\theta\, dr'.$$

Applying the MVT this simplifies to

$$\frac{1}{\pi r^2} \iint_{B(a,r)} u(x,y)\, dx\, dy = \frac{2\, u(a)}{r^2} \int_0^r r'\, dr' = u(a).$$

$|f'(z)|^2$ is a subharmonic function; thus, for all $a \in \mathbb{C}$ and $r > 0$ we have

$$|f'(a)|^2 \leq \frac{1}{\pi r^2} \iint_{B(a,r)} |f'(z)|^2\, dx\, dy.$$

Letting $r \to \infty$ since the integral must be finite, we have that the r^{-2} term forces the inequality $|f'(a)|^2 \leq 0$ which implies that $f'(a) = 0$ everywhere. Thus the function is constant.

(2) Other Problems

Example 2.9.3 (Spring 2023, Problem 5)**.** Suppose that f is an entire function satisfying the functional equation

$$f(f(z)) = c\,f(z) + z\,(1-c)$$

for some fixed $c \neq 1$. Show that $f(z)$ is linear, you may use Picard's theorem.

Solution. Note that by taking the derivative of both sides

$$f(f(z)) = c\,f(z) + z\,(1-c) \implies f'(z)\,f'(f(z)) = c\,f'(z) + 1 - c.$$

If there exists z such that $f'(z) = 0$, then by the above we have that $c = 1$ which immediately yields a contradiction. Thus f is non-constant. Thus by Picard's little theorem we know that $f(\mathbb{C})$ is either $\mathbb{C}$ or $\mathbb{C} \setminus \{a\}$ for some $a \in \mathbb{C}$. However, by the uniformization theorem, we know that there does not exist any conformal map $\mathbb{C} \to \mathbb{C} \setminus \{a\}$. Thus $f(\mathbb{C}) = \mathbb{C}$, and we know the automorphisms of $\mathbb{C}$ are of the form $az + b$.

Example 2.9.4 (Spring 2021, Problem 4)**.** Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic and suppose there is an open set U whose closure $\overline{U} \subset \mathbb{D}$ is in the disk, such that f is injective on U. Must there exist an open set W with $\overline{U} \subset W \subset \mathbb{D}$ such that f is injective on W ? If so, prove your answer, and if not, provide a counterexample. (Here $\mathbb{D}$ is the unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Solution. This is false, consider the following counterexample. Let $a \in \mathbb{D}$, now let $f(z) = (z-a)^2$ and choose the domain $U = \{z \in \mathbb{D} : |z| < |a|\}$. Now note that

$$0 = f'(z) = 2z - 2a \implies z = a.$$

Thus f is injective on U since $a \notin U$; but, f is not injective on $\overline{U}$ since $a \in \overline{U}$. So for $\overline{U} \subset W \subset \mathbb{D}$, f cannot be injective on W because $a \in W$.