Name: ________________________
Email: ________________________

Math Club: Biweekly Contest Week Five

Release Date: April 17, 2024
Instructions: Solve the following problem as best you can. The first student to submit the correct solution via email to tamumathcontest@gmail.com or to Jeremy Kubiak in Blocker 336D (with time stamp) wins!

Problem 1. Show that

$$\sum_{n=3}^{2024} \binom{\binom{n}{2}}{2} \qquad \text{where} \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is divisible by 2022, 2023, and 2026 but not 2024 or 2025.

Solution. Note that

$$\binom{n}{2} = \frac{n(n-1)}{2} \implies \binom{\binom{n}{2}}{2} = \frac{\binom{n}{2}\cdot\left(\binom{n}{2}-1\right)}{2} = \frac{(n-2)(n-1)\cdot n\cdot(n+1)}{8}.$$

This looks something like n choose 4, so we evaluate

$$\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24} \implies 3\cdot\binom{n+1}{4} = \frac{(n-2)(n-1)\cdot n\cdot(n+1)}{8} = \binom{\binom{n}{2}}{2}.$$

Thus

$$\sum_{n=3}^{2024} \binom{\binom{n}{2}}{2} = 3\sum_{n=4}^{2025}\binom{n}{4} = 3\cdot\binom{2026}{5}$$
$$= 3\cdot\frac{2026\cdot 2025\cdot 2024\cdot 2023\cdot 2022}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 2026\cdot 405\cdot 253\cdot 2023\cdot 2022.$$

Clearly this is divisible by 2022, 2023, and 2026. Now suppose that 2024 divides the above. Since 253 divides the above, then it must be true that 8 divides the product

$$2026\cdot 405\cdot 2023\cdot 2022 = 4\cdot 1013\cdot 405\cdot 2023\cdot 1011.$$

But this is clearly a contradiction, thus 2024 can not divide our sum. Similarly suppose that 2025 divides the above. Since 405 divides the above, then it must be true that 5 divides the product

$$2026\cdot 253\cdot 2023\cdot 2022.$$

But this is clearly a contradiction, thus 2025 can not divide our sum.