Name: ________________________________
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Math Club: Biweekly Contest Week Four

Release Date: March 27, 2024
Instructions: Solve the following problem as best you can. The first student to submit the correct solution via email to tamumathcontest@gmail.com or to Jeremy Kubiak in Blocker 336D (with time stamp) wins!

Problem 1. Let $S(n)$ denote the sum of the digits of a positive integer n. Find all solutions to

$$S(n)+S(S(n))+S(S(S(n)))+S(S(S(S(n))))+S(S(S(S(S(n)))))+S(S(S(S(S(S(n))))))=2024.$$

Explain your answer.

Solution. There exists a positive integer m and integers $0 \leq a_k < 10$ such that $n = \sum_{k=0}^{m} a_k \cdot 10^k$. This is merely the base-10 expansion of n. Now note that we have $S(n) = \sum_{k=0}^{m} a_k$. Thus we have

$$n \equiv \sum_{k=0}^{m} a_k \cdot 10^k \equiv \sum_{k=0}^{m} a_k \equiv S(n) \bmod 3.$$

Inductively we have that

$$S(n) \equiv S(S(n)) \equiv \ldots \equiv S(S(S(S(S(S(n)))))) \bmod 3.$$

Suppose there exists a solution n to our equation, then by the above we have that

$$0 \equiv 6 \cdot S(n) \equiv 2024 \equiv 2 \bmod 3.$$

But this is a contradiction; thus, we can not have a solution.