Name: ______________________________
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Math Club: Biweekly Contest Week Three

Release Date: February 28, 2024
Instructions: Solve the following problem as best you can. The first student to submit the correct solution via email to tamumathcontest@gmail.com or to Jeremy Kubiak in Blocker 336D (with time stamp) wins!

Problem 1. Find the solution set to the equation

$$\lfloor x \rfloor^7 + \{x\}^7 = x^7 \qquad \text{where} \qquad \{x\} = x - \lfloor x \rfloor.$$

Solution. Let $x = n + r$ where $n = \lfloor x \rfloor \in \mathbb{Z}$ and $r = \{x\} \in [0, 1)$. Note that

$$\begin{aligned} 0 = x^7 - \lfloor x \rfloor^7 - \{x\}^7 &= (n+r)^7 - n^7 - r^7 \\ &= 7n^6 r + 21n^5 r^2 + 35n^4 r^3 + 35n^3 r^4 + 21n^2 r^5 + 7nr^6 \\ &= 7nr(n+r)(n^2 + nr + r^2)^2. \end{aligned}$$

Thus implies that either $n = 0$, $r = 0$, $n + r = 0$, or $n^2 + nr + r^2 = 0$.

- Note that if $n = 0$ then $r \in [0, 1)$ and $x \in [0, 1)$ is a solution.
- Additionally, if $r = 0$ then $n \in \mathbb{Z}$ and $x \in \mathbb{Z}$ is a solution.
- Note that because $n \in \mathbb{Z}$ and $r \in [0, 1)$ it is impossible for $n + r = 0$ unless $n = 0$ and $r = 0$.
- Note that $n^2 + nr + r^2 = (n + r/2)^2 + 3r^2/4$. Thus if $n^2 + nr + r^2 = 0$ then $n = 0$ and $r = 0$.

So our solution set is $x \in \mathbb{Z} \cup [0, 1)$.