Name: ____________________________________

Email: ____________________________________

Math Club: Biweekly Contest Week Two

Release Date: February 14, 2024

Instructions: Solve the following problem as best you can. The first student to submit the correct solution via email to tamumathcontest@gmail.com or to Jeremy Kubiak in Blocker 336D (with time stamp) wins!

Problem 1. Suppose $a \neq -1$. If $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) + a \cdot f(1 - 1/x) = x$, then show

$$\lim_{x \to \pm\infty} \big(x - (1 + a^3) \cdot f(x)\big) = a.$$

Solution. Let $g(x) = 1 - 1/x$ and note that $(g \circ g \circ g)(x) = x$. Thus note that we have

$$\begin{aligned} f(x) &= x - a \cdot f(g(x)) \\ f(g(x)) &= g(x) - a \cdot f((g \circ g)(x)) \\ f((g \circ g)(x)) &= (g \circ g)(x) - a \cdot f(x). \end{aligned}$$

So we have that

$$\begin{aligned} f(x) = x - a \cdot f(g(x)) &= x - a \cdot (g(x) - a \cdot f((g \circ g)(x))) \\ &= x - a \cdot g(x) + a^2 \cdot f((g \circ g)(x)). \end{aligned}$$

And by extension

$$\begin{aligned} f(x) = x - a \cdot g(x) + a^2 \cdot f((g \circ g)(x)) &= x - a \cdot g(x) + a^2 \cdot ((g \circ g)(x) - a \cdot f(x)) \\ &= x - a \cdot g(x) + a^2 \cdot (g \circ g)(x) - a^3 \cdot f(x). \end{aligned}$$

We conclude that whenever $x \neq 0$ or 1,

$$(1 + a^3)\, f(x) = x - a \cdot g(x) + a^2 \cdot (g \circ g)(x) = x - a \cdot \frac{x - 1}{x} + a^2 \cdot \frac{1}{1 - x}.$$

Thus we can evaluate the limit

$$\lim_{x \to \pm\infty} \big(x - (1 + a^3) \cdot f(x)\big) = \lim_{x \to \pm\infty} \left(a - \frac{a}{x} - \frac{a^2}{1 - x}\right) = a.$$