Name: ______________________________
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Math Club: Biweekly Contest Week One

Release Date: January 31, 2024
Instructions: Solve the following problem as best you can. The first student to submit the correct solution via email to tamumathcontest@gmail.com or to Jeremy Kubiak in Blocker 336D (with time stamp) wins!

Problem 1. Suppose that $x, y,$ and z are positive real numbers such that

$$x^2 + xy + y^2 = 247, \quad y^2 + yz + z^2 = 433, \quad \text{and} \quad z^2 + zx + x^2 = 309.$$

Compute the value of $xy + yz + zx$ and show your work.

Solution. This is secretly a geometry problem. Construct the triangle ABC with the side-lengths $\overline{AB} = \sqrt{247}$, $\overline{BC} = \sqrt{433}$, and $\overline{CA} = \sqrt{309}$. Now choose a point P inside triangle ABC such that $\overline{AP} = x$, $\overline{BP} = y$, and $\overline{CA} = z$. By the law of cosines we have that

$$\begin{aligned}
\cos(\angle APB) &= \frac{x^2 + y^2 - 247}{2xy} = -\frac{xy}{2xy} = -\frac{1}{2} \\
\cos(\angle BPC) &= \frac{y^2 + z^2 - 433}{2yz} = -\frac{yz}{2yz} = -\frac{1}{2} \\
\cos(\angle CPA) &= \frac{z^2 + x^2 - 309}{2zx} = -\frac{zx}{2zx} = -\frac{1}{2}.
\end{aligned}$$

Thus $\angle APB = \angle BPC = \angle CPA = 2\pi/3$. Now using the sine area formula for a triangle we have

$$\begin{aligned}
\text{Area}(\triangle APB) &= \frac{xy}{2} \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{4} xy \\
\text{Area}(\triangle BPC) &= \frac{yz}{2} \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{4} yz \\
\text{Area}(\triangle CPA) &= \frac{zx}{2} \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{4} zx.
\end{aligned}$$

Now we have by Heron's formula

$$\text{Area}(\triangle ABC) = \sqrt{s\,(s-a)(s-b)(s-c)}$$

where $s = (a + b + c)/2$ and $a = \overline{AB}$, $b = \overline{BC}$, and $c = \overline{CA}$. Distributing terms we have

$$\begin{aligned}
\text{Area}(\triangle ABC) &= \frac{1}{4}\sqrt{2\left(a^2b^2 + b^2c^2 + c^2a^2\right) - a^4 - b^4 - c^4} \\
&= \frac{1}{4}\sqrt{2\left(247 \cdot 433 + 433 \cdot 309 + 309 \cdot 247\right) - 247^2 - 433^2 - 309^2} \\
&= \frac{1}{4}\sqrt{2 \cdot 317071 - 61009 - 187489 - 95481} = \frac{1}{4}\sqrt{290163} = \frac{311\sqrt{3}}{4}.
\end{aligned}$$

But also note that

$$\frac{311\sqrt{3}}{4} = \text{Area}(\triangle ABC) = \text{Area}(\triangle APB) + \text{Area}(\triangle BPC) + \text{Area}(\triangle CPA) = \frac{\sqrt{3}}{4}\,(xy + yz + zx).$$

So $xy + yz + zx = 311$.