Name: ______________________________
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Math Club: Biweekly Contest Week Seven

Release Date: November 29, 2023
Instructions: Solve the following problem as best you can. The first student to submit the correct solution via email to tamumathcontest@gmail.com or to Jeremy Kubiak in Blocker 336D (with time stamp) wins!

Problem 1. Using formal power series show that for $|x| < 1$ we have

$$\frac{1}{1-x} = \sum_{0\leq n} x^n \implies \frac{x}{(1-x)^2} = \sum_{0\leq n} nx^n \implies \frac{x^2+x}{(1-x)^3} = \sum_{0\leq n} n^2x^n.$$

Solution. Note that

$$\begin{aligned}\sum_{0\leq n} nx^n &= x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \dots \\ &= x + x^2 + x^3 + x^4 + x^5 + \dots \\ &\quad + x^2 + x^3 + x^4 + x^5 + \dots \\ &\quad + x^3 + x^4 + x^5 + \dots \\ &\quad + x^4 + x^5 + \dots \\ &\quad + x^5 + \dots\end{aligned}$$

Thus we note that

$$\sum_{0\leq n} nx^n = x\sum_{0\leq k}\sum_{0\leq n} x^k x^n = \frac{x}{1-x}\sum_{0\leq k} x^k = \frac{x}{(1-x)^2}.$$

This proves our first implication. Now note that

$$\begin{aligned}\sum_{0\leq n} n^2x^n &= x + 4x^2 + 9x^3 + 16x^4 + 25x^5 + \dots \\ &= x + x^2 + x^3 + x^4 + x^5 + \dots \\ &\quad + 3x^2 + 3x^3 + 3x^4 + 3x^5 + \dots \\ &\quad + 5x^3 + 5x^4 + 5x^5 + \dots \\ &\quad + 7x^4 + 7x^5 + \dots \\ &\quad + 9x^5 + \dots\end{aligned}$$

Thus we note that

$$\begin{aligned}\sum_{0\leq n} n^2x^n = x\sum_{0\leq k}\sum_{0\leq n}(2k+1)x^kx^n &= \frac{x}{1-x}\left(2\sum_{0\leq k} kx^k + \sum_{0\leq k} x^k\right) \\ &= \frac{x}{1-x}\left(\frac{2x}{(1-x)^2} + \frac{1}{1-x}\right) = \frac{x^2+x}{(1-x)^3}.\end{aligned}$$

This proves our second implication.