Name: ____________________
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Math Club: Biweekly Contest Week Six

Release Date: November 15, 2023
Instructions: Solve the following problem as best you can. The first student to submit the correct solution via email to tamumathcontest@gmail.com or to Jeremy Kubiak in Blocker 336D (with time stamp) wins!

Problem 1. Find the integer solution to the following equation which maximizes abc,

$$abc - 9ab - 16bc - 16ca + 144a + 144b + 256c = 2312.$$

Give both the solution (a, b, c) and the corresponding value of abc.

Hint. If $a + b + c = n$ then can you find an upper bound for abc in terms of n? How can you use this to prove your solution maximizes abc?

Solution. First we address the meaning of the hint as this will help us later. Note that by the AM-GM inequality we have that

$$\frac{n}{3} = \frac{a+b+c}{3} \geq \sqrt[3]{abc} \implies \frac{n^3}{27} = \frac{(a+b+c)^3}{27} \geq abc.$$

Thus $abc \leq n^3/27$.

Recall that the AM-GM inequality is actually an equality when $a = b = c$ (important).

Now we solve the main part of the problem. Apply Simon's Favorite Factoring Trick (SFFT) by subtracting 2304 from both sides of the given equation

$$abc - 9ab - 16bc - 16ca + 144a + 144b + 256c - 2304 = (a-16)(b-16)(c-9) = 8.$$

Note that we can factor 8 over $\mathbb{Z}$ into three parts up to permutation as

$$8 = (\pm_1 1) \cdot (\pm_2 1) \cdot (\pm_1(\pm_2 8)), \quad \text{or} \quad (\pm_1 1) \cdot (\pm_2 2) \cdot (\pm_1(\pm_2 4)).$$

Here we have that $\pm_1$ and $\pm_2$ denote independent choices of sign.

Suppose that $a-16$, $b-16$, and $c-9$ are $\pm_1 1$, $\pm_2 1$ and $\pm_1(\pm_2 8)$ up to permutation. Now note,

$$\begin{aligned} a - 16 + b - 16 + c - 9 &= \pm_1 1 \pm_2 1 \pm_1 (\pm_2 8) \\ \implies a + b + c &= 41 \pm_1 1 \pm_2 1 \pm_1 (\pm_2 8) = 33,\ 47,\ 51. \end{aligned}$$

So the sum $a + b + c$ is either 33, 47, or 51 assuming this factorization of 8. Likewise suppose that $a - 16$, $b - 16$, and $c - 9$ are $\pm_1 1$, $\pm_2 2$, and $\pm_1(\pm_2 4)$ up to permutation. Now note,

$$\begin{aligned} a - 16 + b - 16 + c - 9 &= \pm_1 1 \pm_2 2 \pm_1 (\pm_2 4) \\ \implies a + b + c &= 41 \pm_1 1 \pm_2 2 \pm_1 (\pm_2 4) = 36,\ 38,\ 42,\ 48. \end{aligned}$$

So the sum of $a + b + c$ is either 36, 38, 42, or 48 assuming this factorization of 8.

Thus, $a + b + c = 33, 36, 38, 42, 47, 48,$ or 51. And by our previous result $abc \leq 51^3/27 = 4913$.

The final part requires a bit of number sense in recognizing the right choices in finding the maximal solution. Note that if $a + b + c = 51$ then $a - 16$, $b - 16$, and $c - 9$ are 1, 1, and 8 up to permutation. But recalling that the AM-GM inequality is an equality when $a = b = c$ we note that if $a - 16 = 1$, $b - 16 = 1$, and $c - 9 = 8$, then $a = b = c = 17$ and $abc = 4913$ exactly. Thus this must be our maximal solution as we already showed $abc \leq 4913$.