Name: ____________________
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Math Club: Biweekly Contest Week Five

Release Date: November 8, 2023
Instructions: Solve the following problem as best you can. The first student to submit the correct solution via email to tamumathcontest@gmail.com or to Jeremy Kubiak in Blocker 336D (with time stamp) wins!

Problem 1. Let M_2 be the group of invertable 2×2 matrices. Recall that we have both matrix addition and matrix multiplication on M_2, these operations are *commutative* and *non-commutative* respectively. That is, for $a, b \in M_2$ we have that $a + b = b + a$ and $ab \neq ba$. Finally, recall that a matrix times its inverse is the identity matrix. That is for $a \in M_2$ we have that $aa^{-1} = a^{-1}a = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$; additionally, if a matrix is a multiple of the identity matrix we call it a *constant*.

Let $x, y \in M_2$, compute the constant term of the expression $(x + x^{-1} + y + y^{-1})^4$.

Solution. Lets expand out our 4^{th} power,

$$(x + x^{-1} + y + y^{-1})(x + x^{-1} + y + y^{-1})(x + x^{-1} + y + y^{-1})(x + x^{-1} + y + y^{-1}).$$

Note that every term in our product expansion before combining like terms corresponds to a unique choice of 1 element from each of the 4 terms in our product. For example, suppose I choose x from the 1^{st} term, y from the 2^{nd} term, x^{-1} from the 3^{rd} term, and y from the 4^{th} term; this would correspond to the $xyx^{-1}y$ term in our expanded product. Note that we are unable to reduce this to a constant because the matrix product does not commute; however, for a product such as $xyy^{-1}x^{-1}$ or $yy^{-1}x^{-1}x$ we have that

$$xyy^{-1}x^{-1} = x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}x^{-1} = xx^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad yy^{-1}x^{-1}x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We can find all such products of this form which reduce. They are as follows

$$\begin{array}{llll}
xx^{-1}xx^{-1} & xx^{-1}yy^{-1} & xx^{-1}x^{-1}x & xx^{-1}y^{-1}y \\
yy^{-1}xx^{-1} & yy^{-1}yy^{-1} & yy^{-1}x^{-1}x & yy^{-1}y^{-1}y \\
x^{-1}xxx^{-1} & x^{-1}xyy^{-1} & x^{-1}xx^{-1}x & x^{-1}xy^{-1}y \\
y^{-1}yxx^{-1} & y^{-1}yyy^{-1} & y^{-1}yx^{-1}x & y^{-1}yy^{-1}y \\
xxx^{-1}x^{-1} & xyy^{-1}x^{-1} & xy^{-1}yx^{-1} & yxx^{-1}y^{-1} \\
yyy^{-1}y^{-1} & yx^{-1}xy^{-1} & x^{-1}yy^{-1}x & x^{-1}x^{-1}xx \\
x^{-1}y^{-1}yx & y^{-1}xx^{-1}y & y^{-1}x^{-1}xy & y^{-1}y^{-1}yy
\end{array}$$

Note that there are exactly 28 prodcuts which reduce, this is the constant term of the expression.

Comment. More simply put for those with experience in abstract algebra, the constant term is the number of reducible length 4 words on the free group with 2 generators. This begs the more general question, what is the constant term of the expression $(x + y + x^{-1} + y^{-1})^{2n}$.

Solution. Using recursion, we find the generating function with series convergence on $|x| < 1/12$,

$$\frac{3}{1 + 2\sqrt{1 - 12x}} = 1 + 4x + 28x^2 + 232x^3 + 2092x^4 + 19864x^5 + \ldots.$$

This is a deep result, and its solution will be left as an exercise to the reader.