Name: ______________________________
Email: ______________________________

Math Club: Biweekly Contest Week Four

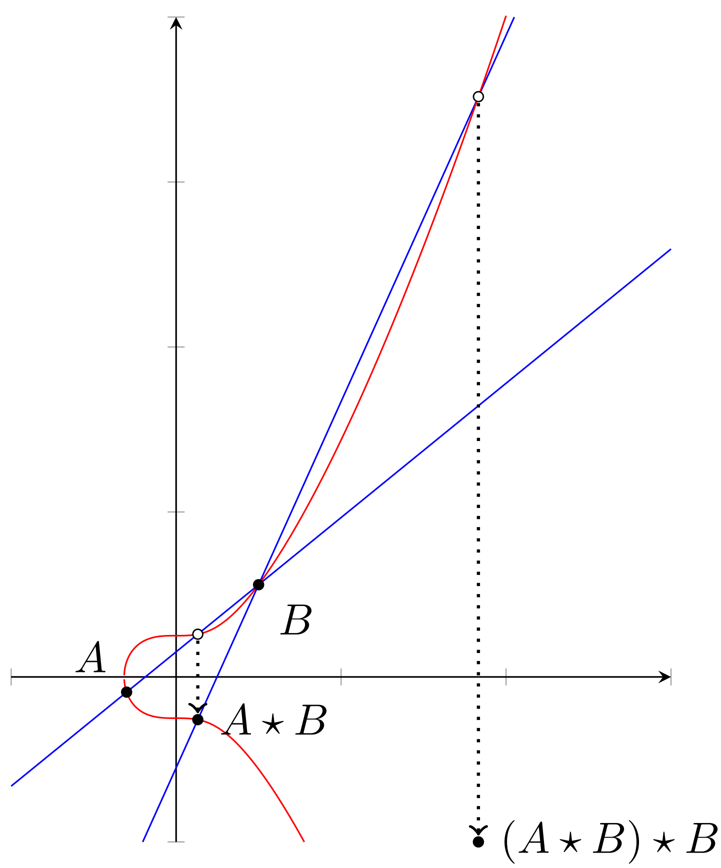
Release Date: October 18, 2023
Instructions: Solve the following problem as best you can. The first student to submit the correct solution via email to tamumathcontest@gmail.com or to Jeremy Kubiak in Blocker 336D (with time stamp) wins!

Problem 1. Let $P = (x_p, y_p)$ and $Q = (x_q, y_q)$ be two solutions to the equation $y^2 = x^3 + 17$ such that $P \neq Q$ and $x_p \neq x_q$. We define an operation $P \star Q$; construct a line interpolating P and Q, this line will intersect $y^2 = x^3 + 17$ at a third point $P \neq R \neq Q$ with $R = (x_r, y_r)$. We let

$$P \star Q = (x_r, -y_r), \quad \text{note this is the reflection of } R \text{ over the } x\text{-axis.}$$

Let $A = (-2, -3)$ and $B = (4, 9)$, compute $C = (A \star B) \star B$.

Hint. Geometrically your construction should look roughly as follows



with $(A \star B) \star B$ lying on the red curve out of range of the displayed graph.

Solution. We compute the line interpolating A and B. Note that this line is of the form $y = \lambda x + \nu$ and A and B are solutions to this equation. This gives us a matrix which can be sent to RREF form to compute λ and ν,

$$\begin{pmatrix} -2 & 1 & -3 \\ 4 & 1 & 9 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \implies \lambda = 2 \quad \text{and} \quad \nu = 1.$$

Now we would like to find the intersection of the line $y = 2x+1$ and the elliptic curve $y^2 = x^3+17$.

$$(2x+1)^2 = x^3 + 17 \implies 4x^2 + 4x + 1 = x^3 + 17 \implies 0 = x^3 - 4x^2 - 4x + 16.$$

Note that by our construction A and B are in the intersection set of $y = 2x + 1$ and $y^2 = x^3 + 17$. So it follows that the x-corrdinates of both A and B should factor from the above cubic,

$$0 = x^3 - 4x^2 - 4x + 16 = (x + 2)(x - 2)(x - 4).$$

Thus our new intersection point has x-coordinate $x = 2$ and y-coordinate $y = 2(2) + 1 = 5$. Reflecting this across the x-axis we have that $A * B = (2, -5)$.

Now we compute the line interpolating $A*B$ and B. Note that this line is of the form $y = \lambda x+\nu$ and A and B are solutions to this equation. This gives us a matrix which can be sent to RREF form to compute λ and ν,

$$\begin{pmatrix} 2 & 1 & -5 \\ 4 & 1 & 9 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & -19 \end{pmatrix} \implies \lambda = 7 \quad \text{and} \quad \nu = -19.$$

Now we would like to find the intersection of the line $y = 7x-19$ and the elliptic curve $y^2 = x^3+17$.

$$(7x - 19)^2 = x^3 + 17 \implies 49x^2 - 266x + 361 = x^3 + 17 \implies 0 = x^3 - 49x^2 + 266x - 344.$$

Note that by our construction $A * B$ and B are both in the intersection set of $y = 7x - 19$ and $y^2 = x^3+17$. So it follows that the x-coordinates of both $A*B$ and B should factor from the above cubic,

$$0 = x^3 - 49x^2 + 266x - 344 = (x - 2)(x - 4)(x - 43).$$

Thus our new intersection point has x-coordinate $x = 43$ and y-coordinate $y = 7(43) + 1 = 282$. Reflecting this across the x-axis we have that $C = (A * B) * B = (43, -282)$.