Name: ______________________________
Email: ______________________________

Math Club: Biweekly Contest Week Two

Release Date: September 13, 2023
Instructions: Solve the following problem as best you can. The first student to submit the correct solution via email to tamumathcontest@gmail.com or to Jeremy Kubiak in Blocker 336D (with time stamp) wins!

Problem 1. Let z be a complex number such that $z^5 = 1$ and $z^4 + z^3 + z^2 + z^1 + 1 = 0$. Suppose, that $z + z^{-1}$ is a root of the integer polynomial $x^2 + bx + c$. Find b and c.

Solution. Since z is a root of the integer polynomial $x^2 + bx + c$ we know that

$$(z + z^{-1})^2 + b\,(z + z^{-1}) + c = z^2 + 2 + z^{-2} + bz + bz^{-1} + c = bz^4 + z^3 + z^2 + bz + c + 2 = 0.$$

Now note that because $z^4 + z^3 + z^2 + z + 1 = 0$ we have that

$$bz^4 + z^3 + z^2 + bz + c + 2 = (b-1)\,z^4 + (b-1)\,z + c + 1 = 0.$$

Note that setting $b = 1$ and $c = -1$ makes this equation true.

However, it is not obvious that this is the only solution; so we will prove this. Note that because $z^5 = 1$ and $z^4 + z^3 + z^2 + z^1 + 1 = 0$ it follows that z is a non-trival power of ζ_5. Thus,

$$z = \cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right) \quad \text{and} \quad z^{-1} = \cos\left(\frac{2k\pi}{5}\right) - i\sin\left(\frac{2k\pi}{5}\right) \quad \text{where } k \in \{1,2,3,4\}.$$

So we have that $z + z^{-1} = 2\cos(2k\pi/5)$. Noting that $5 = 2^{2^1} + 1$ we have that $z + z^{-1}$ is always algebraic of degree 2 (deep result about constructablity and Fermat primes). Thus $z + z^{-1}$ and 1 are linearly independent over $\mathbb{Q}$. Thus we have that,

$$(b-1)\,z^4 + (b-1)\,z + c + 1 = (b-1)(z + z^{-1}) + (c+1)\cdot 1 = 0 \implies (b-1)(z + z^{-1}) = c + 1 = 0.$$

The result follows immediately.