Name: ______________________________
Email: ______________________________

Math Club: Biweekly Contest Week One

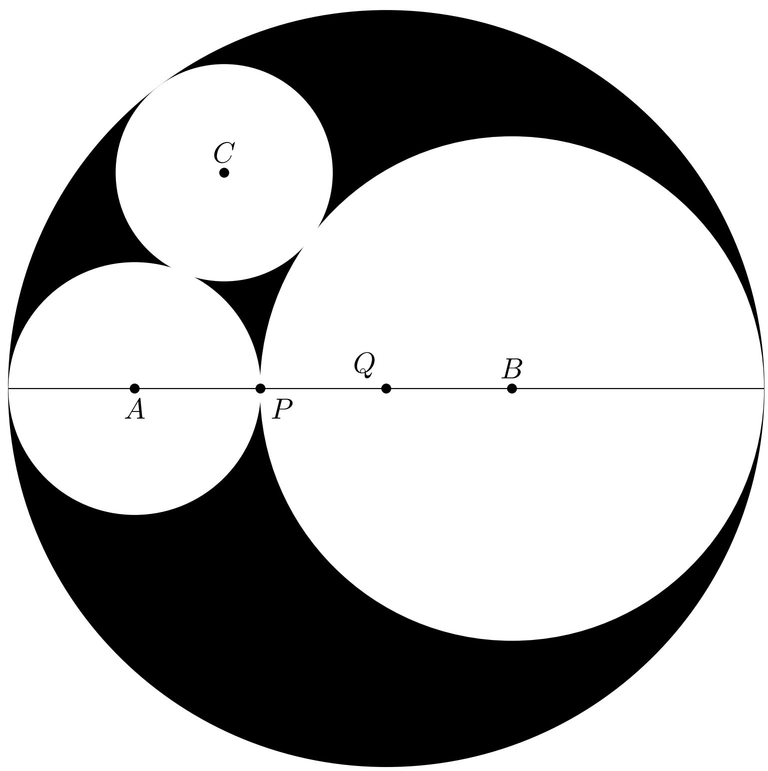
Release Date: August 30, 2023
Instructions: Solve the following problem as best you can. The first student to submit the correct solution via email to tamumathcontest@gmail.com or to Jeremy Kubiak in Blocker 336D (with time stamp) wins!

Problem 1. Place two points A and B such that $|\overline{AB}| = 1$. Now trisect the segment $\overline{AB}$ to create points P and Q. Draw the following three circles:

1. A circle of radius $1/3$ centered at A,
2. A circle of radius $2/3$ centered at B,
3. A circle of radius 1 centered at Q.

A fourth circle; centered at a point C, can be drawn such that it is externally tangent to the circles centered at A and B and internally tangent to the circle centered at Q.

After this construction; fill in the shaded areas as shown. What fraction of the circle centered at Q is shaded?



Solution. This is a simple application of Descartes' Theorem to find the radius of circle C. We have,

$$\left(\frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C} - \frac{1}{r_Q}\right)^2 = \frac{2}{r_A^2} + \frac{2}{r_B^2} + \frac{2}{r_C^2} + \frac{2}{r_Q^2}.$$

Note that the negative sign $-1/r_Q$ indicates the internal tangency of the other circles to Q.

Substituting our values we get

$$\left(3+\frac{3}{2}+\frac{1}{r_C}-1\right)^2 = 18+\frac{9}{2}+\frac{2}{r_C^2}+2 \implies \left(\frac{7}{2}+\frac{1}{r_C}\right)^2 = \frac{49}{4}+\frac{7}{r_C}+\frac{1}{r_C^2} = \frac{49}{2}+\frac{2}{r_C^2}.$$

Finding a common denominator we have that

$$\frac{49r_C^2+28r_C+4}{4r_C^2} = \frac{98r_C^2+8}{4r_C^2} \implies 0 = 49r_C^2-28r_C+4 = (7r_C-2)^2 \implies r_C = 2/7.$$

Now we can compute the fraction of the shaded area as follows,

$$\frac{\pi(r_Q^2-r_A^2-r_B^2-r_C^2)}{\pi r_Q^2} = \frac{\pi(1-1/9-4/9-4/49)}{\pi} = \frac{160}{441}.$$