Name: ______________________________

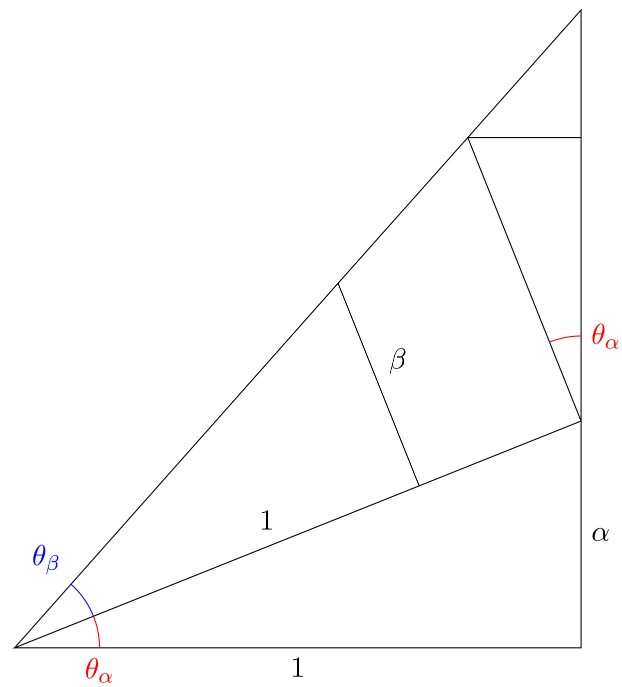
Math Club: Contest Week Six

Release Date: April 19, 2023
Instructions: Solve the following problem the best you can, first to submit the correct solution via email or the secretaries in Room 332 (with time stamp) wins!

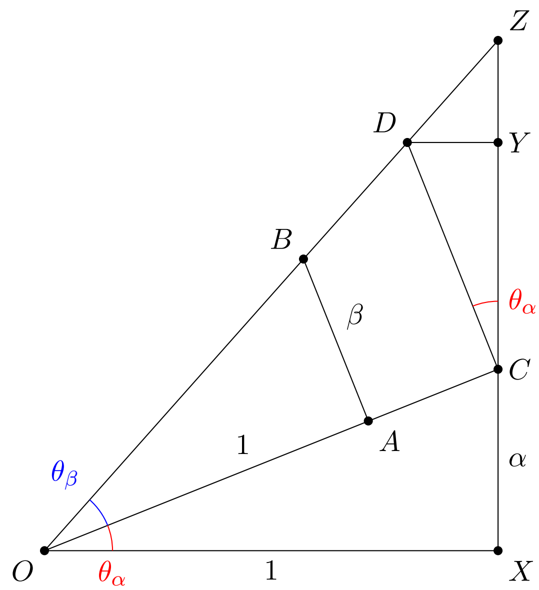
Problem 1. Fill out the side-length of every member in the below diagram to prove that given $\theta_\alpha = \arctan(\alpha)$ and $\theta_\beta = \arctan(\beta)$ respectively, it follows that

$$\theta_\alpha + \theta_\beta = \arctan(\alpha) + \arctan(\beta) = \arctan\left(\frac{\alpha+\beta}{1-\alpha\beta}\right).$$

Some angles and sides have already been filled in for clarity,



We redraw the diagram with labels for reference in solution writing:



Note that because $\triangle OXC$ is right, we can use pythagorean theorem to compute the length of $\overline{OC}$ to be $\sqrt{1+\alpha^2}$. Thus, the length of $\overline{AC}$ is $\sqrt{1+\alpha^2}-1$. Again, noting that $\triangle OAB$ is right, we can use pythagorean theorem to compute the length of $\overline{OB}$ to be $\sqrt{1+\beta^2}$.

Now note that because $\triangle OAB$ and $\triangle OCD$ are similar, we have an implied ratio of sidelengths,

$$\frac{1}{\sqrt{1+\alpha^2}} = \frac{\overline{OA}}{\overline{OC}} = \frac{\overline{AB}}{\overline{CD}} = \frac{\overline{BO}}{\overline{DO}}.$$

Thus, the lengths $\overline{CD}$ and $\overline{DO}$ are $\beta\sqrt{1+\alpha^2}$ and $\sqrt{(1+\alpha^2)(1+\beta^2)}$ respectively. Thus the length $\overline{BD}$ is $\sqrt{1+\beta^2}(\sqrt{1+\alpha^2}-1)$.

Now note that because $\triangle CYD$ and $\triangle OXC$ are similar, we have an implied ratio of sidelengths,

$$\frac{\overline{CY}}{\overline{OX}} = \frac{\overline{YD}}{\overline{XC}} = \frac{\overline{DC}}{\overline{CO}} = \beta.$$

Thus, the lengths $\overline{CY}$ and $\overline{YD}$ are β and $\alpha\beta$ respectively.

Now note that because $\triangle ZYD$ and $\triangle ZXO$ are similar, we have an implied ratio of sidelengths,

$$\alpha\beta = \frac{\overline{YD}}{\overline{XO}} = \frac{\overline{ZY}}{\overline{ZX}} = \frac{\overline{ZY}}{\overline{ZY}+\overline{YC}+\overline{CX}} = \frac{\overline{ZY}}{\overline{ZY}+\beta+\alpha}.$$

Solving this equation we find that the length $\overline{ZY} = (\alpha^2\beta+\alpha\beta^2)/(1-\alpha\beta)$.

Finally, because $\triangle ZYD$ and $\triangle ZXO$ are similar, we have an implied ratio of sidelengths,

$$\alpha\beta = \frac{\overline{YD}}{\overline{XO}} = \frac{\overline{DZ}}{\overline{OZ}} = \frac{\overline{DZ}}{\overline{DZ}+\overline{DO}} = \frac{\overline{DZ}}{\overline{DZ}+\sqrt{(1+\alpha^2)(1+\beta^2)}}.$$

Solving this equation we find that the length $\overline{DZ} = \alpha\beta\sqrt{(1+\alpha^2)(1+\beta^2)}/(1-\alpha\beta)$.

Finally, we note the desired result from the completed diagram,

$$\tan(\theta_\alpha+\theta_\beta) = \frac{\overline{ZX}}{\overline{XO}} = \frac{\overline{ZY}+\overline{YC}+\overline{CX}}{\overline{XO}} = \frac{(\alpha^2\beta+\alpha\beta^2)/(1-\alpha\beta)+\beta+\alpha}{1} = \frac{\alpha+\beta}{1-\alpha\beta}.$$