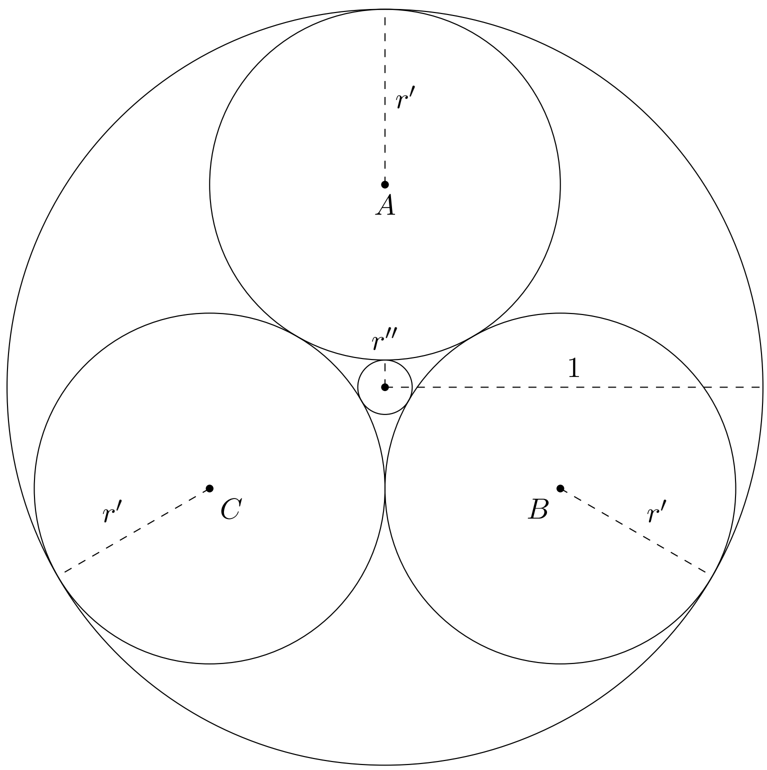
Name: ________________________________

Math Club: Contest Week Three

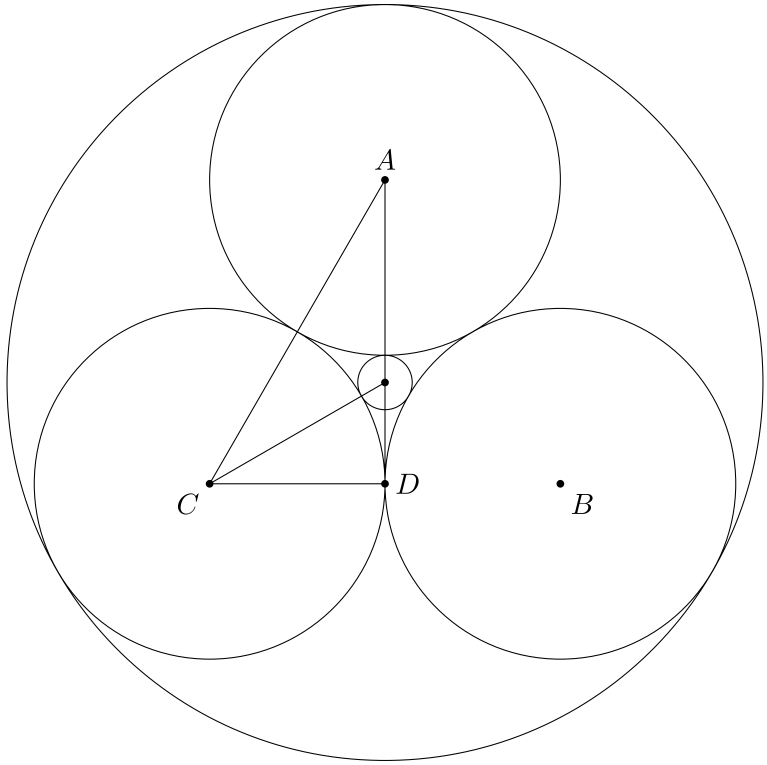
Release Date: February 22, 2023
Instructions: Solve the following problem the best you can, first to submit the correct solution via email or the secretaries in Room 332 (with time stamp) wins!

Problem 1. A circle of radius 1 is centered at the origin. Three circles of radius r' centered at A, B, and C respectively, are constructed inside the radius 1 circle such that there are all tangent to each other and to the radius 1 circle. A smaller, radius r'' circle, is centered at the center of the radius 1 circle such that it is tangent to every radius r' circle. This construction is shown below.



What is r''?

Solution. For clarity of writing, we will refer to the origin as O. Now, note the hidden geometry:



Note that $\overline{OC}$ has length $r' + r''$; thus because $\triangle OCD$ is a $30-60-90$ triangle, we know that $\overline{OD}$ has length $(r'+r'')/2$. Now note that $\overline{AC}$ has length $2r'$; thus because $\triangle ACD$ is a $30-60-90$ triangle, we know that $\overline{AD}$ has length $r'\sqrt{3}$. But we also know that the length of $\overline{AD}$ is the sum of the lengths of $\overline{AO}$ and $\overline{OD}$; thus, the length of $\overline{AD}$ also equals $3(r'+r'')/2$. So by algebraic manipulation we have that $(3-2\sqrt{3})r' + 3r'' = 0$. Trivially, we also have that $2r' + r'' = 1$. This gives us the system of equations

$$\begin{pmatrix} 3-2\sqrt{3} & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} r' \\ r'' \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So, we row reduce the below matrix

$$\begin{pmatrix} 3-2\sqrt{3} & 3 & 0 \\ 2 & 1 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 2\sqrt{3}-3 \\ 0 & 1 & 7-4\sqrt{3} \end{pmatrix}.$$

Thus, $r' = 2\sqrt{3} - 3$ and $r'' = 7 - 4\sqrt{3}$.