Name: ________________________________

Math Club: Contest Week Two

Release Date: February 1, 2023
Instructions: Solve the following problem the best you can, first to submit the correct solution via email or the secretaries in Room 332 (with time stamp) wins!

Problem 1. We define the p-adic valuation $v_p : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$

$$v_p(n) = \max(k \in \mathbb{Z}_{\geq 0} : p^k \text{ divides } n)$$

for all primes p. Make an argument to show that for all $n \in \mathbb{Z}_{>0}$,

$$v_p(1) + v_p(2) + \ldots + v_p(p^n) = \frac{1 - p^n}{1 - p}.$$

Hint. Here are some example p-adic valuations for reference.

- $v_5(375) = 3$ because 5^3 is the greatest power of 5 which divides into 375.
- $v_7(833) = 2$ because 7^2 is the greatest power of 7 which divides into 833.
- $v_{11}(4719) = 2$ because 11^2 is the greatest power of 11 which divides into 4719.

Solution. For solution writing purposes, let $q = p^n$. Also let,

$$N_q(v) = |\{1 \leq k \leq q \mid v_p(k) = v\}|.$$

We will now show using strong induction that,

$$N_q(v-1) = (p-1)\sum_{k=v}^{n} N_q(k).$$

Trivially we have that $N_q(n) = 1$. For our base case we have that $N_q(n-1) = (p-1)N_q(n) = p - 1$ which is trivially true by inclusion-exclusion principle. Now for the strong induction step, we need to show

$$\left(\forall (n > V \geq v),\, N_q(V) = (p-1)\sum_{k=V+1}^{n} N_q(k)\right) \implies \left(N_q(v-1) = (p-1)\sum_{k=v}^{n} N_q(k)\right).$$

Note that via our strong induction hypothesis for all $n - 1 > V \geq v$ it follows that,

$$\begin{aligned} N_q(V) = (p-1)\sum_{k=V+1} N_q(k) &= (p-1)\left(N_q(V+1) + \sum_{k=V+2}^{n} N_q(k)\right) \\ &= p(p-1)\sum_{k=V+2}^{n} N_q(k) \\ &= pN_q(V+1). \end{aligned}$$

By the above and finite summations of geometric series for all $v \leq n$ (note that you have to handle the $v = n$ case separately) it follows that,

$$\sum_{k=v}^{n} N_q(k) = \big(1 + (p-1) + p(1-p) + p^2(1-p) \ldots + p^{n-v-1}(1-p)\big) = 1 - (1 - p^{n-v}) = p^{n-v}$$

Now note that by inclusion-exclusion principle

$$\begin{aligned} N_q(v-1) &= |\{1 \leq k \leq q \mid v_p(k) \geq v-1\}| - \sum_{k=v}^{n} N_q(k) \\ &= qp^{1-v} - \sum_{k=v}^{n} N_q(k) = p^{n-v+1} - p^{n-v} = (p-1)p^{n-v} = (p-1)\sum_{k=v}^{n} N_q(k). \end{aligned}$$

So by strong induction we have shown the desired result and by extension $N_q(v) = p^{n-v} - p^{n-v-1}$ for all $v \leq n$. Now we can evaluate our sum; re-indexing,

$$\begin{aligned} v_p(1) + v_p(2) + \ldots + v_p(q) &= \sum_{k=0}^{n} k \cdot N_q(k) \\ &= 0(p^n - p^{n-1}) + 1(p^{n-1} - p^{n-2}) + \ldots + (n-1)(p^1 - p^0) + n(1) \\ &= p^{n-1} + p^{n-2} + \ldots + p^1 + p^0 = \frac{1-p^n}{1-p}. \end{aligned}$$