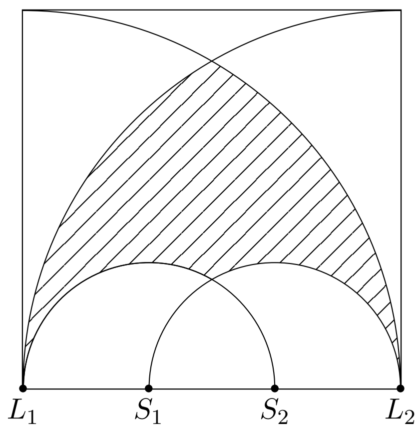
Name: ____________________

Math Club: Contest Week One

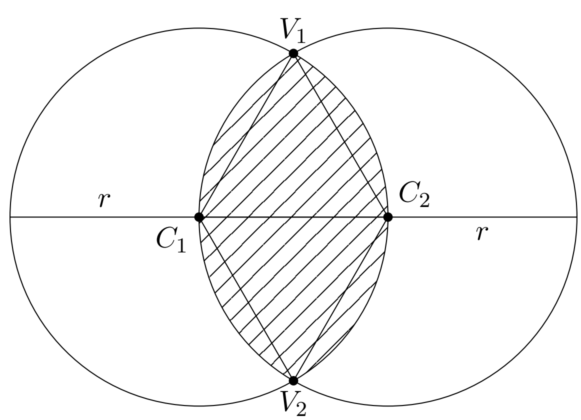
Release Date: January 18, 2023
Instructions: Solve the following problem the best you can, first to submit the correct solution via email or the secretaries in Room 332 (with time stamp) wins!

Problem 1. A square with side length 1 is constructed and two adjacent corners L_1 and L_2 are chosen on this square. The side given by $\overline{L_1L_2}$ is trisected giving two points S_1 and S_2 along $\overline{L_1L_2}$. Arcs of radius 1 are drawn within the square centered at L_1 and L_2. Arcs of radius $1/3$ are drawn within the square centered at S_1 and S_2. This construction is shown below.



Compute the area of the hatched figure.

Solution. Consider two circles of radius r such that the circumference of each passes through the center of the other. We add points and lines to make further argument more clear.



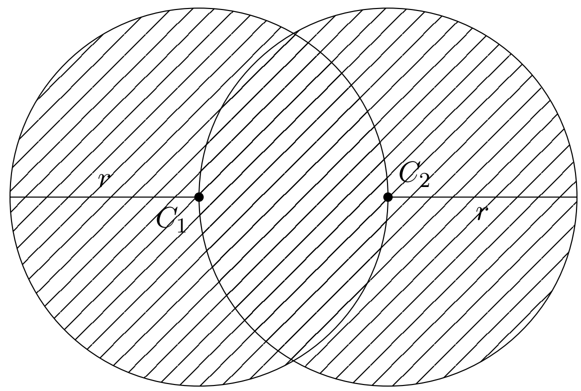
Suppose we wish to find the hatched area. Let T denote the area of the triangles $\triangle V_1C_1C_2$ and $\triangle V_2C_1C_2$ (we know these areas are equal by simple symmetry arguments). Additionally, let S denote the area of the sectors formed by the angles $\angle V_1C_1C_2$, $\angle V_1C_2C_1$, $\angle V_2C_1C_2$, and $\angle V_2C_2C_1$ (again, we know these areas are equal by simple symmetry arguments). We know by the inclusion-exclusion principle that the hatched area is $4S - 2T$. Noting the two equilateral triangles, we get

$$S = \frac{\pi r^2}{6} \qquad \text{and} \qquad T = \frac{r^2\sqrt{3}}{4}.$$

Thus the hatched area in our intermediate exercise is

$$4S - 2T = r^2 \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right).$$

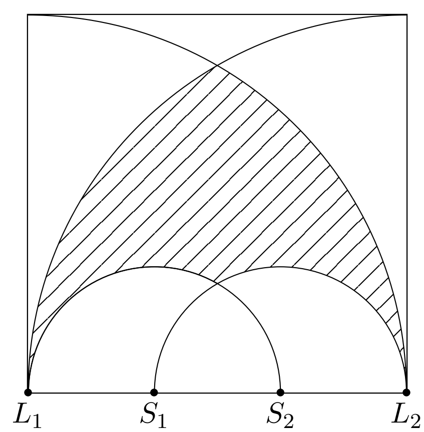
We modify this intermediate exercise slightly.



Suppose we wish to find the hatched area. We know by the inclusion-exclusion principle that the hatched area is

$$\pi r^2 + \pi r^2 - r^2 \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) = r^2 \left(\frac{4\pi}{3} + \frac{\sqrt{3}}{2} \right).$$

Now consider the original figure.



We know by inclusion-exclusion principle and our intermediate results that the hatched area is

$$\frac{1^2}{2} \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) - \frac{(1/3)^2}{2} \left(\frac{4\pi}{3} + \frac{\sqrt{3}}{2} \right) = \frac{7\pi}{27} - \frac{5\sqrt{3}}{18} \approx 0.3334.$$

Solution. Let L_1 be the origin in the plane. Then the circle centered at L_2 has equation

$$(x-1)^2 + y^2 = 1^2 \qquad \text{or} \qquad y = \pm\sqrt{1-(x-1)^2}.$$

Similarly, the circle centered at S_1 has the equation

$$\left(x - \frac{1}{3}\right)^2 + y^2 = \left(\frac{1}{3}\right)^2 \qquad \text{or} \qquad y = \pm\sqrt{\frac{1}{9} - \left(x - \frac{1}{3}\right)^2}.$$

Symmetry arguments and general reasoning allow us to construct an integral for the hatched area

$$2\int_0^{1/2} \sqrt{1-(x-1)^2} - \sqrt{\frac{1}{9} - \left(x - \frac{1}{3}\right)^2}\, dx.$$

We split this into two separate integrals

$$2\left(\int_0^{1/2} \sqrt{1-(x-1)^2}\, dx - \int_0^{1/2} \sqrt{\frac{1}{9} - \left(x - \frac{1}{3}\right)^2}\, dx\right).$$

Each of these integrals is of the form

$$\int \sqrt{a^2 - (x-a)^2}\, dx,$$

for some $a \in \mathbb{R}$, so we wish to evaluate this integral as an intermediate result. Let us use the u-substitution, $u = x - a$. This gives $du = dx$, so

$$\int \sqrt{a^2 - (x-a)^2}\, dx = \int \sqrt{a^2 - u^2}\, du.$$

Now, we do a trigonometric substitution; let $u = a\sin\theta$. This gives $du = a\cos\theta\, d\theta$, so

$$\int \sqrt{a^2 - (x-a)^2}\, dx = \int \sqrt{a^2 - u^2}\, du = \int a^2 \cos^2\theta\, d\theta.$$

Using the identity $2\cos^2\theta = 1 + \cos(2\theta)$, we arrive at the result

$$\int \sqrt{a^2 - (x-a)^2}\, dx = \int \sqrt{a^2 - u^2}\, du = \int a^2 \cos^2\theta\, d\theta = a^2\left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right) + C.$$

Furthermore, as $\sin(2\theta) = 2\sin\theta\cos\theta$ we get

$$\int \sqrt{a^2 - (x-a)^2}\, dx = \int \sqrt{a^2 - u^2}\, du = \int a^2 \cos^2\theta\, d\theta = \frac{a^2}{2}\left(\theta + \sin\theta\cos\theta\right) + C.$$

Now note that because $x - a = u = a\sin\theta$,

$$\begin{aligned} \theta = \sin^{-1}\left(\frac{x-a}{a}\right) \quad &\implies \quad \sin\theta = \frac{x-a}{a} \\ &\implies \quad \cos\theta = \sqrt{1 - \left(\frac{x-a}{a}\right)^2}. \end{aligned}$$

Using this we get that

$$\int \sqrt{a^2 - (x-a)^2}\, dx = \frac{a^2}{2}\left(\sin^{-1}\left(\frac{x-a}{a}\right) + \left(\frac{x-a}{a}\right)\sqrt{1 - \left(\frac{x-a}{a}\right)^2}\right) + C.$$

So,

$$\begin{aligned} \int_0^{1/2} \sqrt{1-(x-1)^2}\, dx &= \frac{1^2}{2}\left(\sin^{-1}\left(-\frac{1}{2}\right) - \frac{1}{2}\sqrt{1 - \left(-\frac{1}{2}\right)^2} - \sin^{-1}(-1) + \sqrt{1-(-1)^2}\right) \\ &= \frac{1}{2}\left(-\frac{\pi}{6} - \frac{1}{2}\sqrt{\frac{3}{4}} + \frac{\pi}{2} + 0\right) = \frac{\pi}{6} - \frac{\sqrt{3}}{8}. \end{aligned}$$

Additionally,

$$\int_0^{1/2} \sqrt{\frac{1}{9} - \left(x - \frac{1}{3}\right)^2}\, dx = \frac{(1/3)^2}{2}\left(\sin^{-1}\left(\frac{1}{2}\right) + \frac{1}{2}\sqrt{1 - \left(\frac{1}{2}\right)^2} - \sin^{-1}(-1) + \sqrt{1 - (-1)^2}\right)$$
$$= \frac{1}{18}\left(\frac{\pi}{6} + \frac{1}{2}\sqrt{\frac{3}{4}} + \frac{\pi}{2} + 0\right) = \frac{\pi}{27} + \frac{\sqrt{3}}{72}.$$

So,

$$2\int_0^{1/2} \sqrt{1 - (x-1)^2} - \sqrt{\frac{1}{9} - \left(x - \frac{1}{3}\right)^2}\, dx = 2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8} - \frac{\pi}{27} - \frac{\sqrt{3}}{72}\right) = \frac{7\pi}{27} - \frac{5\sqrt{3}}{18} \approx 0.3334.$$