Name: ______________________________

Math Club: Contest Week Five

Release Date: November 2, 2022
Instructions: Solve the following problem the best you can, first to submit the correct solution via email or the secretaries in Room 332 (with time stamp) wins!

Problem 1. Let $\Phi_1(x) = x - 1$; otherwise, for prime p, let

$$\Phi_p(x) = \sum_{0 \leq k < p} x^k.$$

More generally, for all n

$$\prod_{d \text{ divides } n} \Phi_d(x) = x^n - 1.$$

Compute $\Phi_{30}(2)$.

Hint: it helps to compute any two of $\Phi_6(x)$, $\Phi_{10}(x)$, or $\Phi_{15}(x)$ before computing $\Phi_{30}(x)$.

Solution. Using the above information we know that

$$x^6 - 1 = \Phi_6(x)\Phi_3(x)\Phi_2(x)\Phi_1(x) = \Phi_6(x)(x^2 + x + 1)(x + 1)(x - 1)$$
$$x^{10} - 1 = \Phi_{10}(x)\Phi_5(x)\Phi_2(x)\Phi_1(x) = \Phi_{10}(x)(x^4 + x^3 + x^2 + x + 1)(x + 1)(x - 1).$$

Thus,

$$x^6 - 1 = \Phi_6(x)(x^4 + x^3 - x - 1) \implies \Phi_6(x) = x^2 - x + 1$$
$$x^{10} - 1 = \Phi_{10}(x)(x^6 + x^5 - x - 1) \implies \Phi_{10}(x) = x^4 - x^3 + x^2 - x + 1$$

by polynomial long division. Using the above information we know that

$$x^{15} - 1 = \Phi_{15}(x)\Phi_5(x)\Phi_3(x)\Phi_1(x)$$
$$x^{30} - 1 = \Phi_{30}(x)\Phi_{15}(x)\Phi_{10}(x)\Phi_6(x)\Phi_5(x)\Phi_3(x)\Phi_2(x)\Phi_1(x).$$

Thus,

$$\begin{aligned} x^{30} - 1 &= \Phi_{30}(x)(x^{15} - 1)\Phi_{10}(x)\Phi_6(x)\Phi_2(x) \\ &= \Phi_{30}(x)(x^{15} - 1)(x^4 - x^3 + x^2 - x + 1)(x^2 - x + 1)(x + 1) \\ &= \Phi_{30}(x)(x^{22} - x^{21} + x^{20} + x^{17} - x^{16} + x^{15} - x^7 + x^6 - x^5 - x^2 + x - 1) \end{aligned}$$

$$\implies \Phi_{30}(x) = x^8 + x^7 - x^5 - x^4 - x^3 + x + 1$$

by polynomial long division. Thus,

$$\Phi_{30}(2) = 2^8 + 2^7 - 2^5 - 2^4 - 2^3 + 2 + 1 = 256 + 128 - 32 - 16 - 8 + 2 + 1 = 331.$$