Name: ____________________

Math Club: Contest Week Four

Release Date: October 19, 2022
Instructions: Solve the following problem the best you can, first to submit the correct solution via email or the secretaries in Room 332 (with time stamp) wins!

Problem 1. If $x+y+z=0$, then compute

$$\left(\frac{y-z}{x}+\frac{z-x}{y}+\frac{x-y}{z}\right)\left(\frac{x}{y-z}+\frac{y}{z-x}+\frac{z}{x-y}\right).$$

Solution. Let us look at

$$\begin{aligned}\frac{y-z}{x}+\frac{z-x}{y}+\frac{x-y}{z} &= \frac{yz(y-z)+zx(z-x)+xy(x-y)}{xyz}\\ &= \frac{y^2z-yz^2+z^2x-x^2z+x^2y-xy^2}{xyz}\\ &= \frac{(z^2+xy-yz-zx)(x-y)}{xyz}\\ &= \frac{(z(z-y)-x(z-y))(x-y)}{xyz} = -\frac{(y-z)(z-x)(x-y)}{xyz}.\end{aligned}$$

Since $x+y+z=0$, it follows that $-x=y+z$, $-y=x+z$, and $-z=x+y$. Thus,

$$\begin{aligned}(z-x)-(x-y) &= z+y-2x=-3x\\ (x-y)-(y-z) &= x+z-2y=-3y\\ (y-z)-(z-x) &= y+x-2z=-3z.\end{aligned}$$

Thus,

$$\frac{x}{y-z}+\frac{y}{z-x}+\frac{z}{x-y} = -\frac{1}{3}\left(\frac{(z-x)-(x-y)}{y-z}+\frac{(x-y)-(y-z)}{z-x}+\frac{(y-z)-(z-x)}{x-y}\right).$$

By substitution reusing the first part of our solution

$$\begin{aligned}\frac{x}{y-z}+\frac{y}{z-x}+\frac{z}{x-y} &= \frac{1}{3}\left(\frac{((y-z)-(z-x))((z-x)-(x-y))((x-y)-(y-z))}{(y-z)(z-x)(x-y)}\right)\\ &= \frac{1}{3}\left(\frac{(-3z)(-3x)(-3y)}{(y-z)(z-x)(x-y)}\right) = -\frac{9xyz}{(y-z)(z-x)(x-y)}.\end{aligned}$$

So

$$\begin{aligned}&\left(\frac{y-z}{x}+\frac{z-x}{y}+\frac{x-y}{z}\right)\left(\frac{x}{y-z}+\frac{y}{z-x}+\frac{z}{x-y}\right)\\ &\qquad = \left(-\frac{(y-z)(z-x)(x-y)}{xyz}\right)\left(-\frac{9xyz}{(y-z)(z-x)(x-y)}\right) = 9.\end{aligned}$$