Name: ______________________________

Math Club: Contest Week Three

Release Date: October 5, 2022
Instructions: Solve the following problem the best you can, first to submit the correct solution via email or the secretaries in Room 332 (with time stamp) wins!

Problem 1. Given

$$\cot^2\left(\frac{\pi}{2n+1}\right)+\cot^2\left(\frac{2\pi}{2n+1}\right)+\ldots+\cot^2\left(\frac{n\pi}{2n+1}\right)=\frac{n(2n-1)}{3}$$

and

$$\csc^2\left(\frac{\pi}{2n+1}\right)+\csc^2\left(\frac{2\pi}{2n+1}\right)+\ldots+\csc^2\left(\frac{n\pi}{2n+1}\right)=\frac{2n(n+1)}{3}.$$

Use the fact that $\cot(x) < x^{-1} < \csc(x)$ when $0 < x \leq \pi/2$ to show that

$$\left(1-\frac{1}{2n+1}\right)\left(1-\frac{2}{2n+1}\right)<\frac{6}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\right)<\left(1-\frac{1}{2n+1}\right)\left(1+\frac{1}{2n+1}\right).$$

Note: after doing so you can then use the squeeze theorem to prove the famous result $\zeta(2)=\pi^2/6$, although this is not required for your submission.

Solution. By the double inequality $\cot(x) < x^{-1} < \csc(x)$ we know

$$\begin{aligned}\cot^2\left(\frac{\pi}{2n+1}\right)+\cot^2\left(\frac{2\pi}{2n+1}\right)+\ldots+\cot^2\left(\frac{n\pi}{2n+1}\right)&\\ &<\left(\frac{2n+1}{\pi}\right)^2+\left(\frac{2n+1}{2\pi}\right)^2+\ldots+\left(\frac{2n+1}{n\pi}\right)^2\\ \frac{n(2n-1)}{3}&<\left(\frac{2n+1}{\pi}\right)^2+\left(\frac{2n+1}{2\pi}\right)^2+\ldots+\left(\frac{2n+1}{n\pi}\right)^2\end{aligned}$$

and

$$\begin{aligned}\csc^2\left(\frac{\pi}{2n+1}\right)+\csc^2\left(\frac{2\pi}{2n+1}\right)+\ldots+\csc^2\left(\frac{n\pi}{2n+1}\right)&\\ &>\left(\frac{2n+1}{\pi}\right)^2+\left(\frac{2n+1}{2\pi}\right)^2+\ldots+\left(\frac{2n+1}{n\pi}\right)^2\\ \frac{2n(n+1)}{3}&>\left(\frac{2n+1}{\pi}\right)^2+\left(\frac{2n+1}{2\pi}\right)^2+\ldots+\left(\frac{2n+1}{n\pi}\right)^2.\end{aligned}$$

Thus

$$\frac{n(2n-1)}{3}<\left(\frac{2n+1}{\pi}\right)^2+\left(\frac{2n+1}{2\pi}\right)^2+\ldots+\left(\frac{2n+1}{n\pi}\right)^2<\frac{2n(n+1)}{3}.$$

We multiply through by $(\pi/(2n+1))^2$ to get

$$\frac{\pi^2}{3}\left(\frac{n(2n-1)}{(2n+1)^2}\right)<\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}<\frac{\pi^2}{3}\left(\frac{2n(n+1)}{(2n+1)^2}\right).$$

Which simplifies to

$$\left(1-\frac{1}{2n+1}\right)\left(1-\frac{2}{2n+1}\right)<\frac{6}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\right)<\left(1-\frac{1}{2n+1}\right)\left(1+\frac{1}{2n+1}\right).$$