Name: _____

Math Club: Contest Week One

Release Date: September 7, 2022
Instructions: Solve the following problem the best you can, first to submit the correct solution via email or the secretaries in Room 332 (with time stamp) wins!

Problem 1. Given

$$x^2 + p_1 x + q_1 = (x - a_1)(x - b_1)$$
$$x^2 + p_2 x + q_2 = (x - a_2)(x - b_2).$$

Rewrite the expression

$$(a_1 - a_2)(a_1 - b_2)(b_1 - a_2)(b_1 - b_2)$$

in terms of $p_1, q_1, p_2,$ and q_2.

Solution. Firstly, note by Vieta's formulae we know

$$p_1 = -a_1 - b_1 \quad q_1 = a_1 b_1$$
$$p_2 = -a_2 - b_2 \quad q_2 = a_2 b_2.$$

These identities will be used heavily in the later stages of this solution. Note, by simple substitution

$$a_1^2 + p_2 a_1 + q_2 = (a_1 - a_2)(a_1 - b_2)$$
$$b_1^2 + p_2 b_1 + q_2 = (b_1 - a_2)(b_1 - b_2).$$

Thus,

$$(a_1 - a_2)(a_1 - b_2)(b_1 - a_2)(b_1 - b_2) = (a_1^2 + p_2 a_1 + q_2)(b_1^2 + p_2 b_1 + q_2).$$

Expanding the right hand side of this expression we get

$$a_1^2 b_1^2 + p_2 a_1^2 b_1 + q_2 a_1^2 + p_2 a_1 b_1^2 + p_2^2 a_1 b_1 + p_2 q_2 a_1 + q_2 b_1^2 + p_2 q_2 b_1 + q_2^2.$$

We can collect and simplify terms (using our previous identities) in the following manner.

$$\begin{aligned}
a_1^2 b_1^2 &= (a_1 b_1)^2 = q_1^2 \\
p_2 a_1^2 b_1 + p_2 a_1 b_1^2 &= -p_2(a_1 b_1)(-a_1 - b_1) = -p_1 p_2 q_1 \\
q_2 a_1^2 + q_2 b_1^2 &= q_2((-a_1 - b_1)^2 - 2(a_1 b_1)) = q_2(p_1^2 - 2q_1) = p_1^2 q_2 - 2q_1 q_2 \\
p_2^2 a_1 b_1 &= p_2^2(a_1 b_1) = p_2^2 q_1 \\
p_2 q_2 a_1 + p_2 q_2 b_1 &= -p_2 q_2(-a_1 - b_1) = -p_1 p_2 q_2 \\
q_2^2 &= q_2^2.
\end{aligned}$$

And so we arrive at the expression

$$q_1^2 - 2q_1 q_2 + q_2^2 + p_1^2 q_2 - p_1 p_2 q_2 + p_2^2 q_1 - p_1 p_2 q_1.$$

Which simplifies as

$$(q_1 - q_2)^2 + (p_1 - p_2)(p_1 q_2 - p_2 q_1).$$