# Fast Computation for Generalized Dedekind Sums

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# Motivation: Classical Dedekind Sum

## Definition

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{SL}_2(\mathbb{Z})$.

$$s(a, c) = \sum_{n=0}^{c-1} B_1\left(\frac{n}{c}\right) B_1\left(\frac{an}{c}\right)$$

where

$$B_1(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Z} \\ x - \lfloor x \rfloor - \frac{1}{2}, & \text{otherwise.} \end{cases}$$

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## Example

$s(5,3) = \sum_{n=0}^{2} B_1\left(\frac{n}{3}\right) B_1\left(\frac{5n}{3}\right) =$

$B_1\left(\frac{0}{3}\right) B_1\left(\frac{0}{3}\right) + B_1\left(\frac{1}{3}\right) B_1\left(\frac{5}{3}\right) + B_1\left(\frac{2}{3}\right) B_1\left(\frac{10}{3}\right) = -\frac{1}{18}$

# Motivation: Computing Classical Dedekind Sum

## Remark

It takes $\mathcal{O}(c)$ time to compute $s(h,k)$ from definition.

# Motivation: Computing Classical Dedekind Sum

## Properties

$$s(a,c) = -s(c,a) + \frac{1}{12}\left(\frac{a}{c} + \frac{1}{ac} + \frac{c}{a}\right) - \frac{1}{4}$$

$$s(a,c) = s(a \text{ mod } c, c)$$

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## Example

$$\begin{aligned} s(99,10) &= s(9,10) \\ &= -s(10,9) + R_1 \\ &= -s(1,9) + R_1 \\ &= s(9,1) + R_1 + R_2 \\ &= s(0,1) + R_1 + R_2 \\ &= R_1 + R_2 \end{aligned}$$

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$$\mathcal{O}(c) \qquad \longrightarrow \qquad \mathcal{O}(\log(c))$$

# Research Question

Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, is there an algorithm to compute the generalized Dedekind sum of $\gamma$ faster than $O(c)$?

# Definitions: Matrix Groups

## Definition

$$\mathsf{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a,\, b,\, c,\, d \in \mathbb{Z};\, ad - bc = 1 \right\}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) : c \equiv 0 \mod N \right\}$$

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## Remark

For

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\mathsf{SL}_2(\mathbb{Z}) = \langle S,\, T \rangle.$$

# Definitions: Dirichlet Characters

## Definition

A *Dirichlet character* $\chi \mod q$ is a function $\chi : \mathbb{Z} \to \mathbb{C}$ with the following properties:

1. $\chi(n + ql) = \chi(n) \; \forall n, l \in \mathbb{Z}$
2. $\chi(n) = 0$ if and only if $\gcd(n, q) \neq 1$
3. $\chi(mn) = \chi(m)\chi(n) \; \forall m, n \in \mathbb{Z}$.

# Generalized Dedekind Sum

## Definition

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q_1q_2)$ with primitive Dirichlet characters $\chi_1, \chi_2$ and respective conductors $q_1, q_2$. Let $q_1, q_2 > 1$ and $\chi_1\chi_2(-1) = 1$, then

$$S_{\chi_1,\chi_2}(\gamma) = \sum_{j=1}^{c} \sum_{i=1}^{q_1} \left( \overline{\chi_2(j)\chi_1(i)} B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right)\right).$$

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## Crossed Homomorphism Property

Let $\gamma_1, \gamma_2 \in \Gamma_0(q_1 q_2)$. Then

$$S_{\chi_1,\chi_2}(\gamma_1\gamma_2) = S_{\chi_1,\chi_2}(\gamma_1) + \psi(\gamma_1) S_{\chi_1,\chi_2}(\gamma_2).$$

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$$S_{\chi_1,\chi_2}(\gamma_1\gamma_2) = S_{\chi_1,\chi_2}(\gamma_1) + \psi(\gamma_1) S_{\chi_1,\chi_2}(\gamma_2).$$

Note that $\psi$ is trivial in $\Gamma_1(N)$, so $S_{\chi_1,\chi_2}$ is a homomorphism from $\Gamma_1(N)$ to $\mathbb{C}$.

# Intuition

Given $\Gamma_1(N) \leq \mathsf{SL}_2(\mathbb{Z})$. Let $\mathsf{SL}_2(\mathbb{Z}) = \langle S_i \rangle$ and $\Gamma_1(N) = \langle \gamma_i \rangle$. Given $\gamma \in \Gamma_1(N)$, we want to find $S_{\chi_1,\chi_2}(\gamma)$.

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$$\begin{aligned} \gamma &= S_1 S_2 \cdots S_m \\ &= \gamma_1 \gamma_2 \cdots \gamma_k. \end{aligned}$$

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$$\begin{aligned} \gamma &= S_1 S_2 \cdots S_m \\ &= \gamma_1 \gamma_2 \cdots \gamma_k. \end{aligned}$$

So

$$S_{\chi_1,\chi_2}(\gamma) = S_{\chi_1,\chi_2}(\gamma_1) + S_{\chi_1,\chi_2}(\gamma_2) + \cdots + S_{\chi_1,\chi_2}(\gamma_k)$$

# General Group Theoretic Preliminaries

## Definition

We say $\mathcal{T}$ is a *right transversal* of $H$ in $G$ if each right coset of $H$ in $G$ contains exactly one element of $\mathcal{T}$. Moreover, $\mathcal{T}$ must contain the identity.

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Given a right transversal $\mathcal{T}$ of $H$ in $G$, a *right coset representative function* for $\mathcal{T}$ is a mapping: $G \to \mathcal{T}$ via $g \mapsto \overline{g}$, where $\overline{g}$ is the unique element in $\mathcal{T}$ such that $Hg = H\overline{g}$.

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## Example

Let $G = \mathbb{Z}, H = 5\mathbb{Z}$, and $G/H = \{0 + 5\mathbb{Z}, 1 + 5\mathbb{Z}, 2 + 5\mathbb{Z}, 3 + 5\mathbb{Z}, 4 + 5\mathbb{Z}\}$.
Let $\mathcal{T} = \{0, 6, 2, 18, -1\}$.
Since $23 \in 3 + 5\mathbb{Z}$, $\overline{23} = 18$.

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## Lemma (Schreier's Lemma)

*Let $\mathcal{S}$ be a set which finitely generates $G$, and let $\mathcal{T}$ be a right transversal of $H$ in $G$. The set*

$$\{U(t,\, s) : \, t \in \mathcal{T},\, s \in \mathcal{S}\}$$

*generates $H$.*

This set is commonly referred to as the *Schreier generators* of $H$.

# General Group Theoretic Preliminaries

## Theorem (Reidemeister Rewriting Process)

*Given a right transversal of $H$ in $G$, let $G = \langle g_1, \cdots, g_n \rangle$. Let $h = g_{q_1}^{\epsilon_1} g_{q_2}^{\epsilon_2} \cdots g_{q_r}^{\epsilon_r} \in H$ (where $\epsilon_k = \pm 1$) be a word in the $g_i$. Define the mapping $\tau$ of the word $h$ by*

$$\tau(h) = U(p_1, g_{q_1})^{\epsilon_1} U(p_2, g_{q_2})^{\epsilon_2} \cdots U(p_r, g_{q_r})^{\epsilon_r},$$

*where*

$$p_k = \begin{cases} \overline{g_{q_1}^{\epsilon_1} g_{q_2}^{\epsilon_2} \cdots g_{q_{k-1}}^{\epsilon_{k-1}}} & \text{if } \epsilon_k = 1 \\ \overline{g_{q_1}^{\epsilon_1} g_{q_2}^{\epsilon_2} \cdots g_{q_k}^{\epsilon_k}} & \text{if } \epsilon_k = -1. \end{cases}$$

*Then $\tau(h) = h$, for all $h \in H$.*

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*Then $\tau(h) = h$, for all $h \in H$.*

## Remark

The Reidemeister rewriting process expresses a word in the generators of $G$ as a word in the Schreier generators of $H$.

# Example: Reidemeister Rewriting Process

Let $G = \langle g_i \rangle$, let $h \in G$ and $h = g_1 g_1 g_1 g_2^{-1}$, then

$$\begin{aligned}
\tau(h) &= U(\overline{1}, g_1)U(\overline{g_1}, g_1)U(\overline{g_1^2}, g_1)U(\overline{g_1^3 g_2^{-1}}, g_2)^{-1} \\
&= \overline{1}g_1(\overline{1g_1})^{-1} \cdot (1g_1)g_1(\overline{g_1g_1})^{-1} \cdot (\overline{g_1g_1})g_1(\overline{g_1g_1g_1})^{-1} \cdot (\overline{g_1g_1g_1})g_2^{-1}(\overline{g_1g_1g_1g_2^{-1}})^{-1} \\
&= \overline{1}g_1(\overline{1g_1})^{-1} \cdot (1g_1)g_1(\overline{g_1g_1})^{-1} \cdot (\overline{g_1g_1})g_1(\overline{g_1g_1g_1})^{-1} \cdot (\overline{g_1g_1g_1})g_2^{-1}(\overline{g_1g_1g_1g_2^{-1}})^{-1} \\
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&= \overline{1}g_1\color{red}{(\overline{1g_1})^{-1} \cdot (1g_1)}g_1\color{red}{(\overline{g_1g_1})^{-1} \cdot (\overline{g_1g_1})}g_1\color{red}{(\overline{g_1g_1g_1})^{-1} \cdot (\overline{g_1g_1g_1})}g_2^{-1}(\overline{g_1g_1g_1g_2^{-1}})^{-1} \\
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\end{aligned}$$

However, this takes a long time!

# Problem

## Lemma

$$SL_2(\mathbb{Z}) = \langle S, T\rangle.$$

*More specifically, one can decompose any matrix $M \in SL_2(\mathbb{Z})$ into the following form:*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm T^{a_1} S T^{a_2} S \dots T^{a_{r-1}} S T^{a_r}.$$

*Note that $-1 = S^2$. Furthermore, we can precisely describe $a_k$ via the Euclidean algorithm.*

## Remark

Note that $r$ grows as $\log(c)$ and $a_1 + a_2 + \ldots + a_r$ grows as $c$.

# General Group Theoretic Preliminaries

## Theorem (Modified Reidemeister Rewriting Process)

*Given a right transversal of $H$ in $G$, let $G = \langle g_1, \cdots, g_n \rangle$. Let $h = g_{q_1}^{a_1} g_{q_2}^{a_2} \cdots g_{q_r}^{a_r} \in H$ (where $a_i \in \mathbb{Z}_{\neq 0}$) be a word in powers of the $g_i$. Define the mapping $\tau$ of the word $h$ by*

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*Then $\tau(h) = h$, for all $h \in H$.*

## Remark

The modified Reidemeister rewriting process expresses a word in the generators of $G$ as a word in specific elements of $H$.

# Example: Modified Reidemeister Rewriting Process

Let $G = \langle g_i \rangle$, let $h \in G$ and $h = g_1^3 g_2^{-1}$. Reidemeister Rewriting Process:

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## Remark

$H$ now has an infinite alphabet.

# Specific Group Theoretic Preliminaries

## Lemma

*Let $a = qN + r$ for $0 \leq r < N$. Let $M \in SL_2(\mathbb{Z})$, then given a right transversal of $\Gamma_1(N)$ in $SL_2(\mathbb{Z})$:*

$$U\big(\overline{M},\, T^a\big) = U^q\big(\overline{M}, T^N\big)U\big(\overline{M},\, T^r\big).$$

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$$S_{\chi_1,\chi_2}\left(U\big(\overline{M}, T^a\big)\right) = qS_{\chi_1,\chi_2}\left(U\big(\overline{M}, T^N\big)\right) + S_{\chi_1,\chi_2}\left(U\big(\overline{M}, T^r\big)\right).$$

# Algorithm

Let $\gamma_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q_1 q_2)$ with primitive Dirichlet characters $\chi_1, \chi_2$ and respective conductors $q_1, q_2$. Let $q_1, q_2 > 1$ and $\chi_1\chi_2(-1) = 1$. We present an algorithm to find $S_{\chi_1,\chi_2}(\gamma_0)$.

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**Group Theoretic Precomputation**

- Find a right transversal $\mathcal{T}_{\Gamma_0}$ of $\Gamma_1(N)$ in $\Gamma_0(N)$.
- Find a right transversal $\mathcal{T}_{\mathsf{SL}_2(\mathbb{Z})}$ of $\Gamma_1(N)$ in $\mathsf{SL}_2(\mathbb{Z})$.
- Find the set $\mathcal{U} = \{U(t, T^i) : t \in \mathcal{T}_{\mathsf{SL}_2(\mathbb{Z})}, 1 \leq i \leq N\} \cup \{U(t, S^k) : t \in \mathcal{T}_{\mathsf{SL}_2(\mathbb{Z})}, 0 \leq k \leq 2\}$.

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**Dedekind Sum Precomputation**

- Compute the Dedekind sums $S_{\chi_1,\chi_2}(\mathcal{T}_{\Gamma_0})$.
- Compute the Dedekind sums $S_{\chi_1,\chi_2}(\mathcal{U})$.

# Algorithm

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$$\tau(\gamma_1) = U(\overline{p_1}, T^{a_1}) U(\overline{p_1 T^{a_1}}, S) U(\overline{p_2}, T^{a_2}) U(\overline{p_2 T^{a_2}}, S) \cdots \\ \cdots U(\overline{p_r}, T^{a_r}) U(\overline{p_r T^{a_r}}, \pm I) = \gamma_1,$$

where

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# Algorithm

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For each exponent of $T$, we write $a_i = q_i N + r_i$ with $0 \le r_i < N$. Then

$$U\big(\overline{p_i}, T^{a_i}\big) = U^{q_i}\big(\overline{p_i}, T^N\big) U\big(\overline{p_i}, T^{r_i}\big).$$

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# Example Computation

Fix $\Gamma_0(9)$. Let $\chi_1 = \chi_2$ be the primitive character modulo $3$ with conductors $q_1 = q_2 = 3$. We want to compute $S_{\chi_1,\chi_2}(\gamma_0)$ where

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$$\begin{aligned} \tau(\gamma_1) = & U(\overline{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}, T^1) U(\overline{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)}, S) U(\overline{\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)}, T^{-2}) U(\overline{\begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}}, S) \cdots \\ & \cdots U(\overline{\begin{pmatrix} -15 & -13 \\ -8 & -7 \end{pmatrix}}, T^{-11}) U(\overline{\begin{pmatrix} -15 & 152 \\ -8 & 81 \end{pmatrix}}, S) U(\overline{\left(\begin{smallmatrix} 152 & 15 \\ 81 & 8 \end{smallmatrix}\right)}, T^{-1}) U(\overline{\begin{pmatrix} 152 & -137 \\ 81 & -73 \end{pmatrix}}, -I) = \gamma_1. \end{aligned}$$

# Example Computation

We expand and simplify the $T$ terms.

$$U\big(\overline{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}, T^1\big) = U^0(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), T^9)U(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), T^1)$$

$$U\big(\overline{\left(\begin{smallmatrix}1&-1\\1&0\end{smallmatrix}\right)}, T^{-2}\big) = U^{-1}(\left(\begin{smallmatrix}1&8\\1&9\end{smallmatrix}\right), T^9)U(\left(\begin{smallmatrix}1&8\\1&9\end{smallmatrix}\right), T^7)$$

$$\vdots$$

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And we simplify the $S$ terms.

$$U\left(\overline{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}, S\right) = U(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), S)$$

$$U\left(\overline{\left(\begin{smallmatrix}1&-3\\1&-2\end{smallmatrix}\right)}, S\right) = U(\left(\begin{smallmatrix}1&6\\1&7\end{smallmatrix}\right), S)$$

$$\vdots$$

$$U\left(\overline{\left(\begin{smallmatrix}-15&152\\-8&81\end{smallmatrix}\right)}, S\right) = U(\left(\begin{smallmatrix}1&8\\1&9\end{smallmatrix}\right), S)$$

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Using this we can express our desired Dedekind sum as a linear combination of precomputed Dedekind sums.

$$\begin{aligned}
S_{\chi_1,\chi_2}(\gamma_1) = {} & 0 \cdot S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),T^9)) + S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),T^1)) + S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),S)) \\
& -1 \cdot S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&8\\1&9\end{smallmatrix}\right),T^9)) + S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&8\\1&9\end{smallmatrix}\right),T^7)) + S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&6\\1&7\end{smallmatrix}\right),S)) \\
& \vdots \\
& -2 \cdot S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right),T^9)) + S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right),T^7)) + S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&8\\1&9\end{smallmatrix}\right),S)) \\
& -1 \cdot S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}8&7\\9&8\end{smallmatrix}\right),T^9)) + S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}8&7\\9&8\end{smallmatrix}\right),T^8)) + S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}8&7\\9&8\end{smallmatrix}\right),S^2)).
\end{aligned}$$

Now, using the precomputed Dedekind sums, we arrive at the final result

$$S_{\chi_1,\chi_2}(\gamma_0) = S_{\chi_1,\chi_2}(\gamma_1) + S_{\chi_1,\chi_2}(g) = 0.$$

# Time Complexity

## Theorem

*Given primitive Dirichlet characters* $\chi_1, \chi_2$ *and respective conductors* $q_1, q_2 > 1$ *such that* $\chi_1\chi_2(-1) = 1$. *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q_1 q_2)$. *For fixed* $q_1, q_2$, *the time complexity of finding* $S_{\chi_1,\chi_2}(\gamma)$ *as a function of* $\gamma$ *is* $O(\log(c))$.

# Time Complexity

Fix $\Gamma_0(9)$. Let $\chi_1 = \chi_2$ be the trivial primitive Dirichlet characters with conductors $q_1 = q_2 = 3$, and let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $c = 9k$, $0 < a < c$, and $\gcd(a, c) = 1$.

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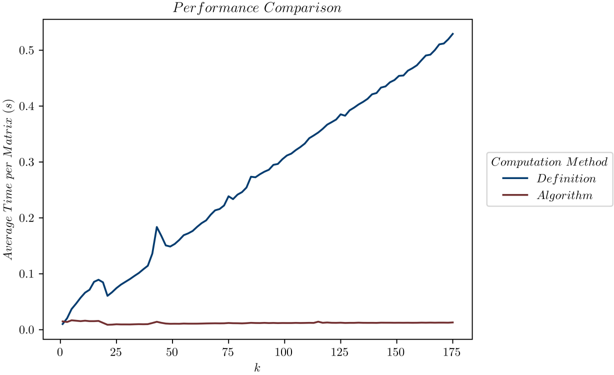

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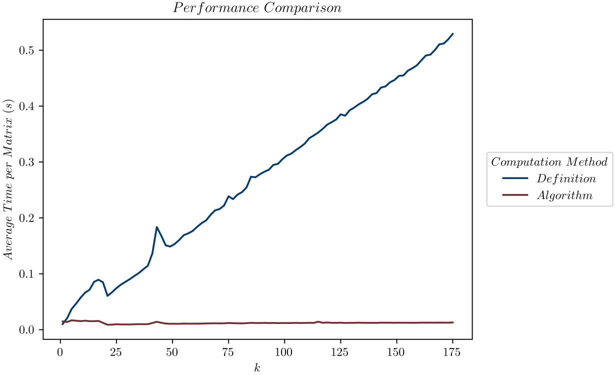


Starting around $k \approx 3$ (thus $c \approx 27$), the performance of the algorithm exceeds that of the definition for this pair of characters.

# Time Complexity

Fix $\Gamma_0(28)$. Let $\chi_1$ be the primitive Dirichlet character with conductor $q_1 = 4$ such that $\chi_1(3) = -1$, and let $\chi_2$ be the primitive Dirichlet character with conductor $q_2 = 7$ such that $\chi_2(3) = \zeta_6^5$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $c = 28k$, $0 < a < c$, and $\gcd(a, c) = 1$.

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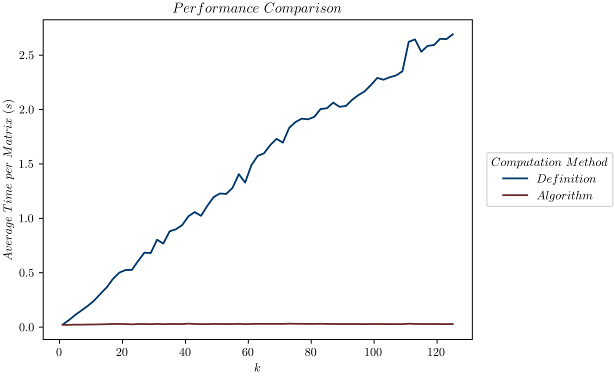

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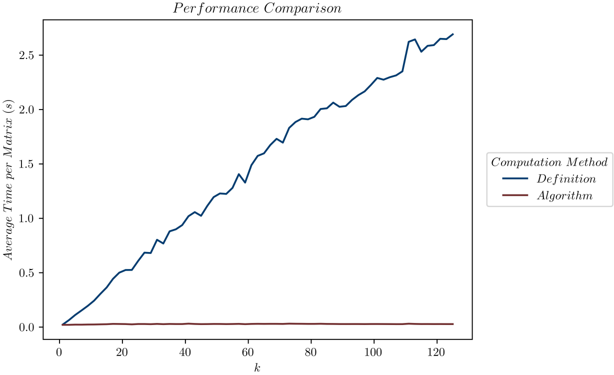


The performance of the algorithm always exceeds that of the definition.

# Conclusion

Thank you for listening!

https://github.com/prestontranbarger/NFDSFastComputation

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