Fast Computation for Generalized Dedekind Sums

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Motivation: Classical Dedekind Sum

Definition

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{SL}_2(\mathbb{Z})$.

$$s(a,c) = \sum_{n=0}^{c-1} B_1\left(\frac{n}{c}\right) B_1\left(\frac{an}{c}\right)$$

where

$$B_1(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Z} \\ x - \lfloor x \rfloor - \frac{1}{2}, & \text{otherwise.} \end{cases}$$

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Example

$s(5,3) = \sum_{n=0}^{2} B_1\left(\frac{n}{3}\right) B_1\left(\frac{5n}{3}\right) =$

$B_1\left(\frac{0}{3}\right) B_1\left(\frac{0}{3}\right) + B_1\left(\frac{1}{3}\right) B_1\left(\frac{5}{3}\right) + B_1\left(\frac{2}{3}\right) B_1\left(\frac{10}{3}\right) = -\frac{1}{18}$

Motivation: Computing Classical Dedekind Sum

Remark

It takes $\mathcal{O}(c)$ time to compute $s(h, k)$ from definition.

Motivation: Computing Classical Dedekind Sum

Properties

$$s(a, c) = -s(c, a) + \frac{1}{12}\left(\frac{a}{c} + \frac{1}{ac} + \frac{c}{a}\right) - \frac{1}{4}$$

$$s(a, c) = s(a \text{ mod } c, c)$$

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Example

$$\begin{aligned} s(99, 10) &= s(9, 10) \\ &= -s(10, 9) + R_1 \\ &= -s(1, 9) + R_1 \\ &= s(9, 1) + R_1 + R_2 \\ &= s(0, 1) + R_1 + R_2 \\ &= R_1 + R_2 \end{aligned}$$

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$$\mathcal{O}(c) \qquad \longrightarrow \qquad \mathcal{O}(\log(c))$$

Research Question

Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, is there an algorithm to compute the generalized Dedekind sum of γ faster than $O(c)$?

Definitions: Matrix Groups

Definition

$$\mathsf{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a,\, b,\, c,\, d \in \mathbb{Z};\, ad - bc = 1 \right\}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) : c \equiv 0 \mod N \right\}$$

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Remark

For

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\mathsf{SL}_2(\mathbb{Z}) = \langle S,\, T \rangle.$$

Definitions: Dirichlet Characters

Definition

A *Dirichlet character* $\chi \mod q$ is a function $\chi : \mathbb{Z} \to \mathbb{C}$ with the following properties:

1. $\chi(n + ql) = \chi(n)\ \forall n, l \in \mathbb{Z}$
2. $\chi(n) = 0$ if and only if $\gcd(n, q) \neq 1$
3. $\chi(mn) = \chi(m)\chi(n)\ \forall m, n \in \mathbb{Z}$.

Generalized Dedekind Sum

Definition

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q_1 q_2)$ with primitive Dirichlet characters χ_1, χ_2 and respective conductors q_1, q_2. Let $q_1, q_2 > 1$ and $\chi_1\chi_2(-1) = 1$, then

$$S_{\chi_1,\chi_2}(\gamma) = \sum_{j=1}^{c}\sum_{i=1}^{q_1}\left(\overline{\chi_2(j)\chi_1(i)}B_1\left(\frac{j}{c}\right)B_1\left(\frac{n}{q_1}+\frac{aj}{c}\right)\right).$$

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Crossed Homomorphism Property

Let $\gamma_1, \gamma_2 \in \Gamma_0(q_1 q_2)$. Then

$$S_{\chi_1,\chi_2}(\gamma_1\gamma_2) = S_{\chi_1,\chi_2}(\gamma_1) + \psi(\gamma_1) S_{\chi_1,\chi_2}(\gamma_2).$$

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$$S_{\chi_1,\chi_2}(\gamma_1\gamma_2) = S_{\chi_1,\chi_2}(\gamma_1) + \psi(\gamma_1) S_{\chi_1,\chi_2}(\gamma_2).$$

Note that ψ is trivial in $\Gamma_1(N)$, so S_{χ_1,χ_2} is a homomorphism from $\Gamma_1(N)$ to $\mathbb{C}$ (more succinctly $S_{\chi_1,\chi_2} \in \mathrm{Hom}(\Gamma_1(N), \mathbb{C})$).

Intuition

Given $\Gamma_1(N) \leq \mathsf{SL}_2(\mathbb{Z})$. Let $\mathsf{SL}_2(\mathbb{Z}) = \langle S_i \rangle$ and $\Gamma_1(N) = \langle \gamma_i \rangle$. Given $\gamma \in \Gamma_1(N)$, we want to find $S_{\chi_1,\chi_2}(\gamma)$.

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$$\begin{aligned} \gamma &= S_1 S_2 \cdots S_m \\ &= \gamma_1 \gamma_2 \cdots \gamma_k. \end{aligned}$$

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$$\begin{aligned} \gamma &= S_1 S_2 \cdots S_m \\ &= \gamma_1 \gamma_2 \cdots \gamma_k. \end{aligned}$$

So

$$S_{\chi_1,\chi_2}(\gamma) = S_{\chi_1,\chi_2}(\gamma_1) + S_{\chi_1,\chi_2}(\gamma_2) + \cdots + S_{\chi_1,\chi_2}(\gamma_k)$$

General Group Theoretic Preliminaries

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We say $\mathcal{T}$ is a *right transversal* of H in G if each right coset of H in G contains exactly one element of $\mathcal{T}$. Moreover, $\mathcal{T}$ must contain the identity.

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Given a right transversal $\mathcal{T}$ of H in G, a *right coset representative function* for $\mathcal{T}$ is a mapping: $G \to \mathcal{T}$ via $g \mapsto \overline{g}$, where $\overline{g}$ is the unique element in $\mathcal{T}$ such that $Hg = H\overline{g}$.

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Example

Let $G = \mathbb{Z}, H = 5\mathbb{Z}$, and $G/H = \{0 + 5\mathbb{Z}, 1 + 5\mathbb{Z}, 2 + 5\mathbb{Z}, 3 + 5\mathbb{Z}, 4 + 5\mathbb{Z}\}$.
Let $\mathcal{T} = \{0, 6, 2, 18, -1\}$.
Since $23 \in 3 + 5\mathbb{Z}$, $\overline{23} = 18$.

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Lemma (Schreier's Lemma)

Let $\mathcal{S}$ be a set which finitely generates G, and let $\mathcal{T}$ be a right transversal of H in G. The set

$$\{U(t, s) : t \in \mathcal{T}, s \in \mathcal{S}\}$$

generates H.

This set is commonly referred to as the *Schreier generators* of H.

General Group Theoretic Preliminaries

Theorem (Reidemeister Rewriting Process)

Given a right transversal of H in G, let $G = \langle g_1, \cdots, g_n \rangle$. Let $h = g_{q_1}^{\epsilon_1} g_{q_2}^{\epsilon_2} \cdots g_{q_r}^{\epsilon_r} \in H$ (where $\epsilon_k = \pm 1$) be a word in the g_i. Define the mapping τ of the word h by

$$\tau(h) = U(p_1, g_{q_1})^{\epsilon_1} U(p_2, g_{q_2})^{\epsilon_2} \cdots U(p_r, g_{q_r})^{\epsilon_r},$$

where

$$p_k = \begin{cases} \overline{g_{q_1}^{\epsilon_1} g_{q_2}^{\epsilon_2} \cdots g_{q_{k-1}}^{\epsilon_{k-1}}} & \text{if } \epsilon_k = 1 \\ \overline{g_{q_1}^{\epsilon_1} g_{q_2}^{\epsilon_2} \cdots g_{q_k}^{\epsilon_k}} & \text{if } \epsilon_k = -1. \end{cases}$$

Then $\tau(h) = h$, for all $h \in H$.

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Then $\tau(h) = h$, for all $h \in H$.

Remark

The Reidemeister rewriting process expresses a word in the generators of G as a word in the Schreier generators of H.

Example: Reidemeister Rewriting Process

Let $G = \langle g_i \rangle$, let $h \in G$ and $h = g_1 g_1 g_1 g_2^{-1}$, then

$$\begin{aligned}
\tau(h) &= U(\overline{1}, g_1)U(\overline{g_1}, g_1)U(\overline{g_1^2}, g_1)U(\overline{g_1^3 g_2^{-1}}, g_2)^{-1} \\
&= \overline{1} g_1 (\overline{1 g_1})^{-1} \cdot (1 g_1) g_1 (\overline{g_1 g_1})^{-1} \cdot (\overline{g_1 g_1}) g_1 (\overline{g_1 g_1 g_1})^{-1} \cdot (\overline{g_1 g_1 g_1}) g_2^{-1} (\overline{g_1 g_1 g_1 g_2^{-1}})^{-1} \\
&= \overline{1} g_1 {\color{red}(\overline{1 g_1})^{-1} \cdot (1 g_1)} g_1 {\color{red}(\overline{g_1 g_1})^{-1} \cdot (\overline{g_1 g_1})} g_1 {\color{red}(\overline{g_1 g_1 g_1})^{-1} \cdot (\overline{g_1 g_1 g_1})} g_2^{-1} (\overline{g_1 g_1 g_1 g_2^{-1}})^{-1} \\
&= \overline{1} g_1 g_1 g_1 g_2^{-1} (\overline{g_1 g_1 g_1 g_2^{-1}})^{-1} \\
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&= g_1 g_1 g_1 g_2^{-1}
\end{aligned}$$

However, this takes a long time!

Problem

Lemma

$$\mathsf{SL}_2(\mathbb{Z}) = \langle S, T \rangle.$$

More specifically, one can decompose any matrix $M \in \mathsf{SL}_2(\mathbb{Z})$ into the following form:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm T^{a_1} S T^{a_2} S \dots T^{a_{r-1}} S T^{a_r}.$$

Note that $-1 = S^2$. Furthermore, we can precisely describe a_k via the Euclidean algorithm.

Remark

Note that $2r$ grows as $\log(c)$ and $|a_1| + |a_2| + \ldots + |a_r| + r$ grows as c.

Theorem (Modified Reidemeister Rewriting Process)

Given a right transversal of H in G, let $G = \langle g_1, \cdots, g_n \rangle$. Let $h = g_{q_1}^{a_1} g_{q_2}^{a_2} \cdots g_{q_r}^{a_r} \in H$ (where $a_i \in \mathbb{Z}_{\neq 0}$) be a word in powers of the g_i. Define the mapping τ of the word h by

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where

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Then $\tau(h) = h$, for all $h \in H$.

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Then $\tau(h) = h$, for all $h \in H$.

Remark

The modified Reidemeister rewriting process expresses a word in the generators of G as a word in specific elements of H.

Example: Modified Reidemeister Rewriting Process

Let $G = \langle g_i \rangle$, let $h \in G$ and $h = g_1^3 g_2^{-1}$. Reidemeister Rewriting Process:

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Modified Reidemeister Rewriting Process:

$$\tau(h) = U(\overline{1}, g_1^3)U(\overline{g_1^3}, g_2^{-1})$$

Remark

H now has an infinite alphabet.

Specific Group Theoretic Preliminaries

Lemma

Let $a = qN + r$ for $0 \leq r < N$. Let $M \in SL_2(\mathbb{Z})$, then given a right transversal of $\Gamma_1(N)$ in $SL_2(\mathbb{Z})$:

$$U\big(\overline{M}, T^a\big) = U^q\big(\overline{M}, T^N\big)U\big(\overline{M}, T^r\big).$$

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$$S_{\chi_1,\,\chi_2}\left(U\big(\overline{M},\, T^a\big)\right) = qS_{\chi_1,\,\chi_2}\left(U\big(\overline{M}, T^N\big)\right) + S_{\chi_1,\,\chi_2}\left(U\big(\overline{M}, T^r\big)\right).$$

Algorithm

Let $\gamma_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q_1 q_2)$ with primitive Dirichlet characters χ_1, χ_2 and respective conductors q_1, q_2. Let $q_1, q_2 > 1$ and $\chi_1\chi_2(-1) = 1$. We present an algorithm to find $S_{\chi_1,\chi_2}(\gamma_0)$.

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Group Theoretic Precomputation

- Find a right transversal $\mathcal{T}_{\Gamma_0}$ of $\Gamma_1(N)$ in $\Gamma_0(N)$.
- Find a right transversal $\mathcal{T}_{\mathsf{SL}_2(\mathbb{Z})}$ of $\Gamma_1(N)$ in $\mathsf{SL}_2(\mathbb{Z})$.
- Find the set $\mathcal{U} = \{U(t, T^i) : t \in \mathcal{T}_{\mathsf{SL}_2(\mathbb{Z})}, 1 \leq i \leq N\} \cup \{U(t, S^k) : t \in \mathcal{T}_{\mathsf{SL}_2(\mathbb{Z})}, 0 \leq k \leq 2\}$.

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Dedekind Sum Precomputation

- Compute the Dedekind sums $S_{\chi_1,\chi_2}(\mathcal{T}_{\Gamma_0})$.
- Compute the Dedekind sums $S_{\chi_1,\chi_2}(\mathcal{U})$.

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The Main Computation

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$$\tau(\gamma_1) = U(\overline{p_1}, T^{a_1}) U(\overline{p_1 T^{a_1}}, S) U(\overline{p_2}, T^{a_2}) U(\overline{p_2 T^{a_2}}, S) \cdots$$
$$\cdots U(\overline{p_r}, T^{a_r}) U(\overline{p_r T^{a_r}}, \pm I) = \gamma_1,$$

where

$$p_k = T^{a_1} S T^{a_2} S \ldots T^{a_{k-1}} S.$$

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$$\gamma_1 = \pm T^{a_1} S T^{a_2} S \dots T^{a_{r-1}} S T^{a_r}.$$

$$\tau(\gamma_1) = U(\overline{p_1}, T^{a_1}) U(\overline{p_1 T^{a_1}}, S) U(\overline{p_2}, T^{a_2}) U(\overline{p_2 T^{a_2}}, S) \cdots$$
$$\cdots U(\overline{p_r}, T^{a_r}) U(\overline{p_r T^{a_r}}, \pm I) = \gamma_1,$$

where

$$p_k = T^{a_1} S T^{a_2} S \dots T^{a_{k-1}} S.$$

For each exponent of T, we write $a_i = q_i N + r_i$ with $0 \leq r_i < N$. Then

$$U\big(\overline{p_i}, T^{a_i}\big) = U^{q_i}\big(\overline{p_i}, T^N\big) U\big(\overline{p_i},\, T^{r_i}\big).$$

Algorithm

The Main Computation

$$\gamma_0 = \gamma_1 g.$$

$$\gamma_1 = \pm T^{a_1} S T^{a_2} S \dots T^{a_{r-1}} S T^{a_r}.$$

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For each exponent of T, we write $a_i = q_i N + r_i$ with $0 \le r_i < N$. Then

$$U\left(\overline{p_i}, T^{a_i}\right) = U^{q_i}\left(\overline{p_i}, T^N\right) U\left(\overline{p_i}, T^{r_i}\right).$$

$$S_{\chi_1,\chi_2}\left(U\left(\overline{p_i}, T^{a}\right)\right) = q_i S_{\chi_1,\chi_2}\left(U\left(\overline{p_i}, T^{N}\right)\right) + S_{\chi_1,\chi_2}\left(U\left(\overline{p_i}, T^{r_i}\right)\right).$$

Example Computation

Fix $\Gamma_0(9)$. Let $\chi_1 = \chi_2$ be the primitive character modulo 3 with conductors $q_1 = q_2 = 3$. We want to compute $S_{\chi_1,\chi_2}(\gamma_0)$ where

$$\gamma_0 = \begin{pmatrix} 17 & 32 \\ 9 & 17 \end{pmatrix}.$$

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$$\begin{aligned}\tau(\gamma_1) =& U(\overline{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}, T^1) U(\overline{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)}, S) U(\overline{\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)}, T^{-2}) U(\overline{\left(\begin{smallmatrix} 1 & -3 \\ 1 & -2 \end{smallmatrix}\right)}, S) \cdots \\ & \cdots U(\overline{\left(\begin{smallmatrix} -15 & -13 \\ -8 & -7 \end{smallmatrix}\right)}, T^{-11}) U(\overline{\left(\begin{smallmatrix} -15 & 152 \\ -8 & 81 \end{smallmatrix}\right)}, S) U(\overline{\left(\begin{smallmatrix} 152 & 15 \\ 81 & 8 \end{smallmatrix}\right)}, T^{-1}) U(\overline{\left(\begin{smallmatrix} 152 & -137 \\ 81 & -73 \end{smallmatrix}\right)}, -I) = \gamma_1.\end{aligned}$$

Example Computation

We expand and simplify the T terms.

$$U\left(\overline{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)},T^{1}\right) = U^{0}((\begin{smallmatrix}1&0\\0&1\end{smallmatrix}),T^{9})U((\begin{smallmatrix}1&0\\0&1\end{smallmatrix}),T^{1})$$

$$U\left(\overline{\left(\begin{smallmatrix}1&-1\\1&0\end{smallmatrix}\right)},T^{-2}\right) = U^{-1}((\begin{smallmatrix}1&8\\1&9\end{smallmatrix}),T^{9})U((\begin{smallmatrix}1&8\\1&9\end{smallmatrix}),T^{7})$$

$$\vdots$$

$$U\left(\overline{\left(\begin{smallmatrix}-15&-13\\-8&-7\end{smallmatrix}\right)},T^{-11}\right) = U^{-2}((\begin{smallmatrix}1&1\\1&2\end{smallmatrix}),T^{9})U((\begin{smallmatrix}1&1\\1&2\end{smallmatrix}),T^{7})$$

$$U\left(\overline{\left(\begin{smallmatrix}152&15\\81&8\end{smallmatrix}\right)},T^{-1}\right) = U^{-1}((\begin{smallmatrix}8&7\\9&8\end{smallmatrix}),T^{9})U((\begin{smallmatrix}8&7\\9&8\end{smallmatrix}),T^{8})$$

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And we simplify the S terms.

$$U\left(\overline{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)},S\right)=U(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),S)$$

$$U\left(\overline{\left(\begin{smallmatrix}1&-3\\1&-2\end{smallmatrix}\right)},S\right)=U(\left(\begin{smallmatrix}1&6\\1&7\end{smallmatrix}\right),S)$$

$$\vdots$$

$$U\left(\overline{\left(\begin{smallmatrix}-15&152\\-8&81\end{smallmatrix}\right)},S\right)=U(\left(\begin{smallmatrix}1&8\\1&9\end{smallmatrix}\right),S)$$

$$U\left(\overline{\left(\begin{smallmatrix}152&-137\\81&-73\end{smallmatrix}\right)},-I\right)=U(\left(\begin{smallmatrix}8&7\\9&8\end{smallmatrix}\right),S^2).$$

Using this we can express our desired Dedekind sum as a linear combination of precomputed Dedekind sums.

$$\begin{aligned} S_{\chi_1,\chi_2}(\gamma_1) = &\ 0 \cdot S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), T^9)) + S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), T^1)) + S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), S)) \\ &-1 \cdot S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&8\\1&9\end{smallmatrix}\right), T^9)) + S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&8\\1&9\end{smallmatrix}\right), T^7)) + S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&6\\1&7\end{smallmatrix}\right), S)) \\ &\vdots \\ &-2 \cdot S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right), T^9)) + S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right), T^7)) + S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}1&8\\1&9\end{smallmatrix}\right), S)) \\ &-1 \cdot S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}8&7\\9&8\end{smallmatrix}\right), T^9)) + S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}8&7\\9&8\end{smallmatrix}\right), T^8)) + S_{\chi_1,\chi_2}(U(\left(\begin{smallmatrix}8&7\\9&8\end{smallmatrix}\right), S^2)). \end{aligned}$$

Now, using the precomputed Dedekind sums, we arrive at the final result

$$S_{\chi_1,\chi_2}(\gamma_0) = S_{\chi_1,\chi_2}(\gamma_1) + S_{\chi_1,\chi_2}(g) = 0.$$

Time Complexity

Theorem

Given primitive Dirichlet characters χ_1, χ_2 *and respective conductors* $q_1, q_2 > 1$ *such that* $\chi_1\chi_2(-1) = 1$. *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q_1 q_2)$. *For fixed* q_1, q_2, *the time complexity of finding* $S_{\chi_1,\chi_2}(\gamma)$ *as a function of* γ *is* $O(\log(c))$.

Experimental Results

Consider $\Gamma_0(28)$. Let χ_1 be the primitive Dirichlet character with conductor $q_1 = 4$, and let χ_2 be the primitive Dirichlet character with conductor $q_2 = 7$ such that $\chi_2(3) = \exp(2\pi i(5/6))$. We let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $c = 28k$, $0 < a < c$, and $\gcd(a, c) = 1$. We choose b and d such that the exponent a_r is 0. We graph the logarithm of the average time it take to compute $S_{\chi_1,\chi_2}(\gamma)$ for all γ as a function of k.

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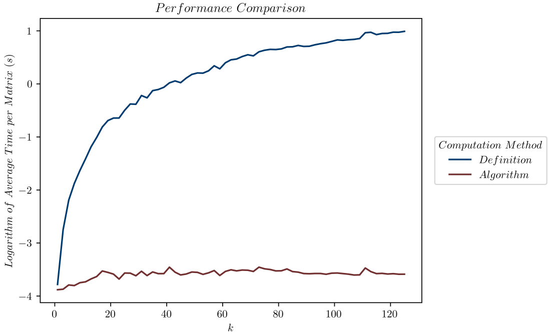

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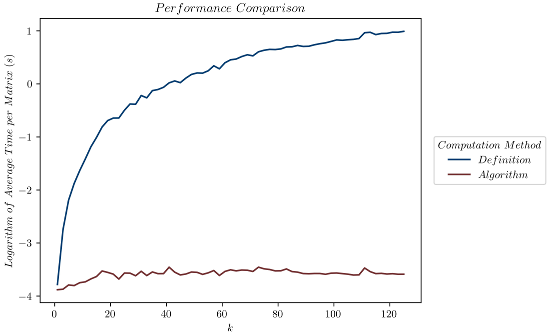

The performance of the algorithm always exceeds that of the definition.

Experimental Results

Now we present an example for a large matrix. Consider $\Gamma_0(35)$. Let χ_1 be the primitive Dirichlet character with conductor $q_1 = 5$ such that $\chi_1(2) = -i$, and let χ_2 be the primitive Dirichlet character with conductor $q_2 = 7$ such that $\chi_2(3) = \exp(2\pi i(1/3))$.

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$$\gamma = \begin{pmatrix} 46741638 & 43234369 \\ 43234205 & 39990117 \end{pmatrix}.$$

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$$\gamma = \begin{pmatrix} 46741638 & 43234369 \\ 43234205 & 39990117 \end{pmatrix}.$$

Computing $S_{\chi_1,\chi_2}(\gamma)$ by definition takes $5.531 * 10^4$ seconds (around 15 hours), whereas it takes $5.128 * 10^{-2}$ seconds using our algorithm.

Conclusion

Thank you for listening!

https://arxiv.org/abs/2210.01172

https://github.com/prestontranbarger/NFDSFastComputation

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