Generalized Dedekind Sums Arising from Specialized Eichler-Shimura Type Integrals

Preston Tranbarger, Texas A&M University

Introduction

Definition 1 ([2], Theorem 4.5.1)**.** *The Fourier expansion for holomorphic weight k Eisenstein series attached to primitive non-trivial Dirichlet characters χ_1 and χ_2 such that $\chi_1\chi_2(-1) = (-1)^k$ is given as*

$$E_{\chi_1,\chi_2,k}(z) = \sum_{1\le N}\sum_{A|N} \chi_1(A)\overline{\chi_2}(N/A)(N/A)^{k-1}e(Nz) \quad \text{where} \quad e(z) = \exp(2\pi i z).$$

The central character of this series is $\psi = \chi_1\overline{\chi_2}$. Given a matrix $\gamma \in \Gamma_0(q_1q_2)$ recall that we have the relation under Mobius transformations given by $E_{\chi_1,\chi_2,k}(\gamma z) = \psi(d_\gamma)\, j(\gamma, z)^k\, E_{\chi_1,\chi_2,k}(z)$.

Definition 2. *Let us define the polynomial*

$$P_{k-2}(z) = (Xz+Y)^{k-2} = \left(\begin{pmatrix} X & Y\end{pmatrix}\begin{pmatrix} z \\ 1\end{pmatrix}\right)^{k-2}.$$

Given a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ we have the relation under Mobius transformations given by

$$P_{k-2}(\gamma z) = (X\gamma z + Y)^{k-2} = \left(j(\gamma,z)^{-1}\begin{pmatrix} X & Y\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} z \\ 1\end{pmatrix}\right)^{k-2}.$$

For fixed z, this polynomial $P_{k-2}(z)$ is a member of $V_{k-2}(\mathbb{C})$ which is the vector space of degree $k-2$ homogeneous polynomials in two variables having complex coefficients.

Definition 3. *Given a function $f: \mathcal{H} \to \mathbb{C}$ and a matrix $\gamma \in SL_2(\mathbb{Z})$, we define the weight k slash operator as $f|_\gamma(z) = j(\gamma,z)^{-k}f(\gamma z)$.*

Definition 4. *Given weight k, primitive non-trivial Dirichlet characters χ_1 and χ_2 with conductors q_1 and q_2 respectively such that $\chi_1\chi_2(-1) = (-1)^k$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(q_1q_2)$, and $z_0 = -d/c + i/(c^2u)$ such that $\gamma z_0 = a/c + iu$; we define $\phi_{\chi_1,\chi_2,k}(\gamma, P_{k-2})$ as,*

$$\phi_{\chi_1,\chi_2,k}(\gamma, P_{k-2}) = \int_{\infty}^{\gamma\infty} E_{\chi_1,\chi_2,k}(z)P_{k-2}(z)\,dz = \lim_{u\to 0^+}\int_{z_0}^{\gamma z_0} E_{\chi_1,\chi_2,k}(z)P_{k-2}(z)\,dz.$$

By specializing X and Y such that $P_{k-2}(z) = (z-a/c)^{k-2}$ we define another function,

$$S_{\chi_1,\chi_2,k}(\gamma) = (-1)^k\tau(\overline{\chi_1})(k-1)\phi_{\chi_1,\chi_2,k}\left(\gamma, (z-a/c)^{k-2}\right).$$

Theorem 1. *For $k \ge 3$, given primitive non-trivial Dirichlet characters χ_1 and χ_2 with conductors q_1 and q_2 respectively such that $\chi_1\chi_2(-1) = (-1)^k$; if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(q_1q_2)$, then*

$$S_{\chi_1,\chi_2,k}(\gamma) = \sum_{\substack{j \bmod c \\ n \bmod q_1}} \overline{\chi_1}(n)\overline{\chi_2}(j)B_1\left(\frac{j}{c}\right)B_{k-1}\left(\frac{aj}{c}+\frac{n}{q_1}\right).$$

Homological Properties

Definition 5. *Recall that $V_{k-2}(\mathbb{C})$ is the vector space of degree $k-2$ homogeneous polynomials in two variables having complex coefficients. Let $P(X,Y)$ be a polynomial in $V_{k-2}(\mathbb{C})$; given a central character ψ, we define $V_{k-2}^{\psi}(\mathbb{C})$ as the same vector space of polynomials with the right group action $V_{k-2}(\mathbb{C}) \times \Gamma_0(q_1q_2) \to V_{k-2}(\mathbb{C})$ via the map*

$$P(X,Y)\cdot\begin{pmatrix} a & b \\ c & d\end{pmatrix} \mapsto \psi(d)\,P\left(\begin{pmatrix} X & Y\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right).$$

It is simple to verify that this indeed forms a right group action. Note that in the case of $P_{k-2}(z) \in V_{k-2}^{\psi}(\mathbb{C})$ one has the relation $P_{k-2}(z)\cdot\gamma = \psi(d_\gamma)P|_\gamma(z)$.

Lemma 1. *Suppose we have two matrices $\gamma_1, \gamma_2 \in \Gamma_0(q_1q_2)$. We have that $\phi_{\chi_1,\chi_2,k}$ is a crossed homomorphism*

$$\phi_{\chi_1,\chi_2,k}(\gamma_1\gamma_2, P_{k-2}) = \phi_{\chi_1,\chi_2,k}(\gamma_1, P_{k-2}) + \phi_{\chi_1,\chi_2,k}(\gamma_2, P_{k-2}\cdot\gamma_1).$$

Corollary 1. *We recover the homological properties of $S_{\chi_1,\chi_2,2}$ as in [3] Lemma 2.2. Specifically we can view $S_{\chi_1,\chi_2,2}$ as an element of the space $Hom(\Gamma_1(q_1q_2),\mathbb{C})$ and we have a crossed homomorphism relation for $S_{\chi_1,\chi_2,2}$ given by*

$$S_{\chi_1,\chi_2,2}(\gamma_1\gamma_2) = S_{\chi_1,\chi_2,2}(\gamma_1) + \psi(d_{\gamma_1})S_{\chi_1,\chi_2,2}(\gamma_2).$$

Analysis

Lemma 2. *For an integer $K \ge 1$ and $z \in \mathcal{H}$ we have*

$$\sum_{1\le B}\overline{\chi_2}(B)B^Ke^{2\pi ABiz} = 2q_2^K\left(\frac{\tau(\overline{\chi_2})\,K!}{(-2\pi i)^{K+1}}\right)\sum_{v\in\mathbb{Z}}\frac{\chi_2(v)}{(Az+v)^{K+1}}.$$

Lemma 3. *For $K > 0$, when $c \nmid A$ we have that $\sum_{v\in\mathbb{Z}}|Aa/c+v|^{-1-K} < \infty$.*

Lemma 4. *For an integer $K \ge 1$, when $c \nmid A$ we have that*

$$\left|\lim_{u\to0^+}\sum_{1\le B}\overline{\chi_2}(B)B^Ke^{2\pi AB(ia/c-u)}\right| = \left|2q_2^K\left(\frac{\tau(\overline{\chi_2})\,K!}{(-2\pi i)^{K+1}}\right)\sum_{v\in\mathbb{Z}}\frac{\chi_2(v)}{(Aa/c+v)^{K+1}}\right| < \infty.$$

Corollary 2. *For an integer $K \ge 1$, when $c \nmid A$ we have that*

$$\lim_{u\to0^+}\sqrt{u}\sum_{1\le B}\overline{\chi_2}(B)B^Ke^{2\pi AB(ia/c-u)} = 0.$$

Lemma 5. *For integers $k \ge 3$ and $0 \le n < k-2$ we have*

$$\lim_{u\to0^+}\sqrt{u}\sum_{1\le A}\frac{\chi_1(A)}{A^{n+1}}\sum_{1\le B}\overline{\chi_2}(B)B^{k-n-2}e^{2\pi AB(ia/c-u)} = 0.$$

Corollary 3. *For integers $k \ge 3$ and $0 \le n < k-2$ we have*

$$\lim_{u\to0^+}u^{k-n-2}\sum_{1\le A}\frac{\chi_1(A)}{A^{n+1}}\sum_{1\le B}\overline{\chi_2}(B)B^{k-n-2}e^{2\pi AB(ia/c-u)} = 0.$$

Lemma 6. *For $k \ge 3$, we can interchange the sum and limit*

$$\lim_{u\to0^+}\sum_{1\le A}\frac{\chi_1(A)}{A^{k-1}}\sum_{1\le B}\overline{\chi_2}(B)e^{2\pi AB(ia/c-u)} = \sum_{1\le A}\frac{\chi_1(A)}{A^{k-1}}\lim_{u\to0^+}\sum_{1\le B}\overline{\chi_2}(B)e^{2\pi AB(ia/c-u)}.$$

Theorem 1 Proof Sketch

We seek to simplify our complicated Eichler-Shimura type integral to a finite sum; this process roughly follows [3]. We begin by substituting Definition 1 into Definition 4,

$$\phi_{\chi_1,\chi_2,k}(\gamma, P_{k-2}) = 2\lim_{u\to0^+}\int_{z_0}^{\gamma z_0}\left((Xz+Y)^{k-2}\sum_{1\le N}\sum_{A|N}\chi_1(A)\overline{\chi_2}(N/A)(N/A)^{k-1}e(Nz)\right)dz.$$

We may interchange the sums and integral due to the rapid decay of $e(Nz)$ when $\mathfrak{Im}(z) > 0$; thus

$$\phi_{\chi_1,\chi_2,k}(\gamma, P_{k-2}) = 2\lim_{u\to0^+}\sum_{1\le N}\sum_{A|N}\int_{z_0}^{\gamma z_0}(Xz+Y)^{k-2}\chi_1(A)\overline{\chi_2}(N/A)(N/A)^{k-1}e(Nz)\,dz.$$

Using repeated integration by parts differentiating $(Xz+Y)^{k-2}$ and integrating $e(Nz)$, we have

$$\phi_{\chi_1,\chi_2,k}(\gamma, P_{k-2}) = 2\lim_{u\to0^+}\sum_{1\le N}\sum_{A|N}\sum_{n=0}^{k-2}\left(\frac{\chi_1(A)\overline{\chi_2}(N/A)N^{k-n-2}}{-A^{k-1}(-2\pi i)^{n+1}}\left(\frac{d^n}{dz^n}(Xz+Y)^{k-2}\right)e(Nz)\right)\Bigg|_{z_0}^{\gamma z_0}.$$

These terms evaluated at z_0 vanish in the limit, thus we are left with $\phi_{\chi_1,\chi_2,k}(\gamma, P_{k-2}) =$

$$2\lim_{u\to0^+}\sum_{1\le N}\sum_{A|N}\sum_{n=0}^{k-2}\left(\frac{\chi_1(A)\overline{\chi_2}(N/A)N^{k-n-2}}{-A^{k-1}(-2\pi i)^{n+1}}\right)\left(\frac{X^n(k-2)!}{(k-n-2)!}(X\gamma z_0+Y)^{k-n-2}e(N\gamma z_0)\right).$$

Specializing X and Y so $P_{k-2}(z) = (z-a/c)^{k-2}$; we have $(-1)^kS_{\chi_1,\chi_2,k}(\gamma)/(\tau(\overline{\chi_1})(k-1)) =$

$$\sum_{n=0}^{k-2}\left(-\frac{2i^{k-n-2}(k-2)!}{(-2\pi i)^{n+1}(k-n-2)!}\right)\left(\lim_{u\to0^+}u^{k-n-2}\sum_{1\le A}\frac{\chi_1(A)}{A^{n+1}}\sum_{1\le B}\overline{\chi_2}(B)B^{k-n-2}e^{2\pi AB(ia/c-u)}\right).$$

Applying Corollary 3 and Lemma 6 we have

$$\frac{(-1)^kS_{\chi_1,\chi_2,k}(\gamma)}{\tau(\overline{\chi_1})(k-1)} = \frac{(k-2)!}{\pi i}\left(-\frac{1}{2\pi i}\right)^{k-2}\sum_{1\le A}\frac{\chi_1(A)}{A^{k-1}}\lim_{u\to0^+}\sum_{1\le B}\overline{\chi_2}(B)e^{2\pi AB(ia/c-u)}.$$

Theorem 1 Proof Sketch (Cont'd)

Using [3] Corollary 3.3 gives

$$\frac{(-1)^kS_{\chi_1,\chi_2,k}(\gamma)}{\tau(\overline{\chi_1})(k-1)} = -\frac{(k-2)!}{\pi i}\left(-\frac{1}{2\pi i}\right)^{k-2}\sum_{1\le A}\sum_{0\le j<c}\left(\frac{\chi_1(A)\overline{\chi_2}(j)}{A^{k-1}}\right)B_1\left(\frac{j}{c}\right)e_c(Aaj).$$

Applying clever re-indexing tricks to this sum gives that $(-1)^kS_{\chi_1,\chi_2,k}(\gamma) = \chi_2(-1)\overline{S}_{\overline{\chi_1},\overline{\chi_2},k}(\gamma)$. So,

$$S_{\chi_1,\chi_2,k}(\gamma) = \frac{1}{2}\left(S_{\chi_1,\chi_2,k}(\gamma) + \chi_1(-1)\overline{S}_{\overline{\chi_1},\overline{\chi_2},k}(\gamma)\right).$$

Thus we have,

$$\begin{aligned}\frac{(-1)^kS_{\chi_1,\chi_2,k}(\gamma)}{\tau(\overline{\chi_1})(k-1)} = &-\left(\frac{(k-2)!}{2\pi i}\right)\left(-\frac{1}{2\pi i}\right)^{k-2}\sum_{1\le A}\sum_{0\le j<c}\left(\frac{\chi_1(A)\overline{\chi_2}(j)}{A^{k-1}}\right)B_1\left(\frac{j}{c}\right)e_c(Aaj)\\ &+\chi_1(-1)\left(\frac{(k-2)!}{2\pi i}\right)\left(\frac{1}{2\pi i}\right)^{k-2}\sum_{1\le A}\sum_{0\le j<c}\left(\frac{\chi_1(A)\overline{\chi_2}(j)}{A^{k-1}}\right)B_1\left(\frac{j}{c}\right)e_c(-Aaj).\end{aligned}$$

Re-indexing and swapping the order of summation we have

$$\frac{(-1)^kS_{\chi_1,\chi_2,k}(\gamma)}{\tau(\overline{\chi_1})(k-1)} = -\left(\frac{(k-2)!}{2\pi i}\right)\left(-\frac{1}{2\pi i}\right)^{k-2}\sum_{0\le j<c}\overline{\chi_2}(j)B_1\left(\frac{j}{c}\right)\sum_{A\neq0}\frac{\chi_1(A)}{A^{k-1}}e_c(Aaj).$$

[1] Definiton 1 gives

$$B_{k-1,\chi}(x) = \begin{cases} \dfrac{i^k\tau(\overline{\chi})(k-1)!}{m(2\pi/m)^{k-1}}\displaystyle\sum_{n\neq0}\frac{\chi(n)\sin(2\pi nx/m)}{n^{k-1}} & \text{if } k \text{ even and } \chi \text{ even,}\\ -\dfrac{i^k\tau(\overline{\chi})(k-1)!}{m(2\pi/m)^{k-1}}\displaystyle\sum_{n\neq0}\frac{\chi(n)i\cos(2\pi nx/m)}{n^{k-1}} & \text{if } k \text{ even and } \chi \text{ odd,}\\ \dfrac{i^k\tau(\overline{\chi})(k-1)!}{m(2\pi/m)^{k-1}}\displaystyle\sum_{n\neq0}\frac{\chi(n)i\cos(2\pi nx/m)}{n^{k-1}} & \text{if } k \text{ odd and } \chi \text{ even,}\\ -\dfrac{i^k\tau(\overline{\chi})(k-1)!}{m(2\pi/m)^{k-1}}\displaystyle\sum_{n\neq0}\frac{\chi(n)\sin(2\pi nx/m)}{n^{k-1}} & \text{if } k \text{ odd and } \chi \text{ odd,}\end{cases}$$

where m is the modulus of χ. We can express this more simply as

$$B_{k-1,\chi}(x) = \frac{(-i)^k\tau(\overline{\chi})(k-1)!}{im(2\pi/m)^{k-1}}\sum_{n\neq0}\frac{\chi(n)}{n^{k-1}}e_m(nx).$$

Substituting this into our expression we have that $(-1)^kS_{\chi_1,\chi_2,k}(\gamma)/(\tau(\overline{\chi_1})(k-1)) =$

$$-\left(\frac{(k-2)!}{2\pi i}\right)\left(-\frac{1}{2\pi i}\right)^{k-2}\left(\frac{iq_1(2\pi/q_1)^{k-1}}{(-i)^k\tau(\overline{\chi_1})(k-1)!}\right)\sum_{0\le j<c}\overline{\chi_2}(j)B_1\left(\frac{j}{c}\right)B_{k-1,\chi_1}\left(\frac{ajq_1}{c}\right).$$

Now applying [1] Theorem 3.1 gives us an equivalent expression $(-1)^kS_{\chi_1,\chi_2,k}(\gamma)/(\tau(\overline{\chi_1})(k-1)) =$

$$-\left(\frac{(k-2)!}{2\pi i}\right)\left(-\frac{1}{2\pi i}\right)^{k-2}\left(\frac{iq_1(2\pi/q_1)^{k-1}}{(-i)^k\tau(\overline{\chi_1})(k-1)!}\right)q_1^{k-2}\sum_{\substack{j \bmod c \\ n \bmod q_1}}\overline{\chi_1}(n)\overline{\chi_2}(j)B_1\left(\frac{j}{c}\right)B_{k-1}\left(\frac{aj}{c}+\frac{n}{q_1}\right)$$

Simplifying the leading coefficient gives

$$\frac{(-1)^kS_{\chi_1,\chi_2,k}(\gamma)}{\tau(\overline{\chi_1})(k-1)} = \frac{(-1)^k}{\tau(\overline{\chi_1})(k-1)}\sum_{\substack{j \bmod c \\ n \bmod q_1}}\overline{\chi_1}(n)\overline{\chi_2}(j)B_1\left(\frac{j}{c}\right)B_{k-1}\left(\frac{aj}{c}+\frac{n}{q_1}\right).$$

Removing the scalar factor from both sides proves the theorem.

References

[1] Bruce C Berndt. "Character Analogues of the Poisson and Euler-MacLaurin Summation Formulas with Applications". In: *Journal of Number Theory* 7.4 (1975), pp. 413–445. ISSN: 0022-314X. DOI: `https://doi.org/10.1016/0022-314X(75)90045-1`.

[2] Fred Diamond and Jerry Shurman. *A First Course in Modular Forms*. Vol. 228. Graduate Texts in Mathematics. Springer-Verlag, New York, 2005, pp. xvi+436. ISBN: 0-387-23229-X.

[3] T. Stucker, A. Vennos, and M. P. Young. "Dedekind Sums Arising from Newform Eisenstein Series". In: *Int. J. Number Theory* 16.10 (2020), pp. 2129–2139. ISSN: 1793-0421. DOI: `10.1142/S1793042120501092`. URL: `https://doi.org/10.1142/S1793042120501092`.