Fast Computation of Generalized Dedekind Sums

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Introduction

The generalized Dedekind sum below is first derived in [2].

Definition 1. *Let χ_1 and χ_2 be primitive Dirichlet characters with respective conductors q_1 and q_2 greater than 1 such that $\chi_1\chi_2(-1) = 1$, and let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q_1q_2)$.*

$$S_{\chi_1,\chi_2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_{j=1}^{c}\sum_{i=1}^{q_1}\left(\overline{\chi_2(j)\chi_1(i)}B_1\Big(\frac{j}{c}\Big)B_1\Big(\frac{n}{q_1} + \frac{aj}{c}\Big)\right).$$

This generalized Dedekind sum exhibits a particularly useful crossed homomorphism property.

Lemma 1. *Let $\gamma_1, \gamma_2 \in \Gamma_0(q_1q_2)$. Then*

$$S_{\chi_1,\chi_2}(\gamma_1\gamma_2) = S(\gamma_1) + \psi(\gamma_1)S(\gamma_2)$$

Remark. *Note $\psi(\gamma)$ is trivial if $\gamma \in \Gamma_1(q_1q_2)$, so S_{χ_1,χ_2} may be viewed as being an element of* $\mathrm{Hom}(\Gamma_1(q_1q_2), \mathbb{C})$.

General Preliminaries

In this section we will define some general group theoretic definitions and results which will aid in the construction of the algorithm. For the rest of this subsection, we let G be a finitely generated group and H be a subgroup of G. We begin by defining right transversals and some associated notation.

Definition 2. *We say $\mathcal{T}$ is a* right transversal *of H in G if each right coset of H in G contains exactly one element of $\mathcal{T}$. Moreover, $\mathcal{T}$ must contain the identity.*

Definition 3. *Given a right transversal $\mathcal{T}$ of H in G, a* right coset representative function *for $\mathcal{T}$ is a mapping: $G \to \mathcal{T}$ via $g \mapsto \overline{g}$, where $\overline{g}$ is the unique element in $\mathcal{T}$ such that $Hg = H\overline{g}$.*

We define a function which plays a critical role in our algorithm.

Definition 4. *Given a right transversal of H in G and $a, b \in G$, we define*

$$U(a, b) = ab(\overline{ab})^{-1}.$$

Using this we define a rewriting process.

Theorem 1 (Modified Reidemeister Rewriting Process)**.** *Given a right transversal of H in G, let $G = \langle g_1, \cdots, g_n\rangle$. Let $h = g_{q_1}^{a_1}g_{q_2}^{a_2}\cdots g_{q_r}^{a_r} \in H$ (where $a_i \in \mathbb{Z}_{\neq 0}$) be a word in powers of the g_i. Define the mapping τ of the word h by*

$$\tau(h) = U(p_1, g_{q_1}^{a_1})U(p_2, g_{q_2}^{a_2})\cdots U(p_r, g_{q_r}^{a_r}),$$

where

$$p_k = \overline{g_{q_1}^{a_1}g_{q_2}^{a_2}\cdots g_{q_{k-1}}^{a_{k-1}}}.$$

Then $\tau(h) = h$, for all $h \in H$.

Specific Preliminaries

Let us now consider the subgroup $\Gamma_1(N)$ of $\mathrm{SL}_2(\mathbb{Z})$.

Definition 5. *Let*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The following lemma is well known.

Lemma 2 (Matrix Decomposition in $\mathrm{SL}_2(\mathbb{Z})$ [1, Theorem 1.2.4])**.** *We know*

$$SL_2(\mathbb{Z}) = \langle S, T\rangle.$$

More specifically, one can decompose any matrix $M \in SL_2(\mathbb{Z})$ into the following form:

$$M = \pm T^{a_1}ST^{a_2}S\ldots T^{a_{k-1}}ST^{a_k}.$$

Note that $-I = S^2$.

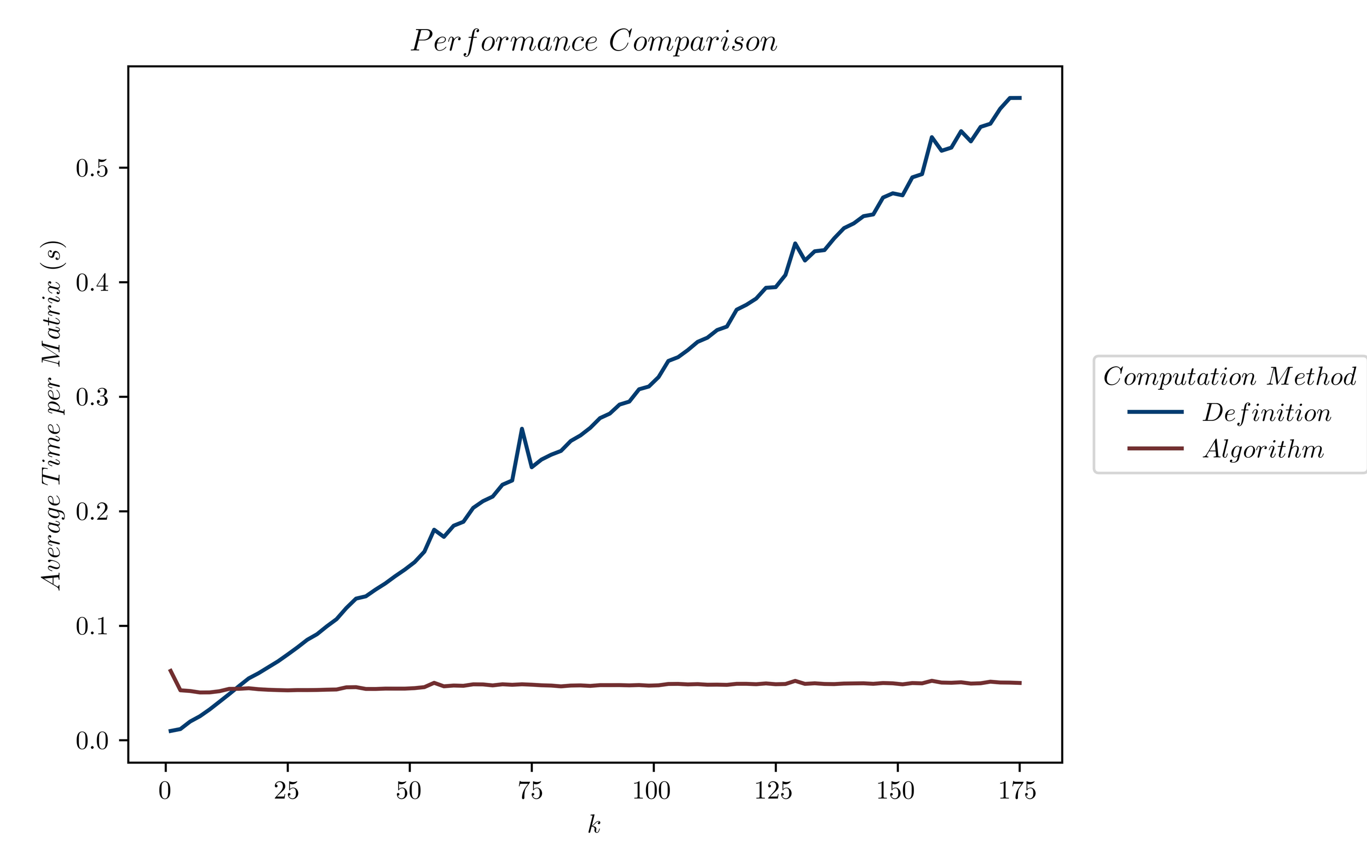
We develop a property of U-functions on the $\Gamma_1(N)$ congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Lemma 3. *Let $a = qN + r$ for $0 \le r < N$ and let $M \in SL_2(\mathbb{Z})$. Given a right transversal of $\Gamma_1(N)$ in $SL_2(\mathbb{Z})$,*

$$U(\overline{M}, T^a) = U^q(\overline{M}, T^N)U(\overline{M}, T^r).$$

Performance Comparison

We compare the computational complexity of our algorithm against that of simply using Definition 1. Fixing χ_1 and χ_2 as the primitive Dirichlet character modulo 3 (that is to say $q_1 = q_2 = 3$ and $N = 9$), we compute the average time per matrix required to compute all $S_{\chi_1,\chi_2}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Where $c = Nk$, $0 < a < c$, $\gcd(a, c) = 1$, and choose b and d such that $a_k = 0$ after applying Lemma 2.



From the graph it is clear that the performance of the algorithm far exceeds that of the definition. This illustrates that not only does this algorithm achieve a theoretical improvement over the definition, but its implementation is practical even for relatively small matrices (for this particular set of characters this threshold is around $c \approx 150$).

The Algorithm

Precomputation

First, we find a right transversal $\mathcal{T}_{\Gamma_0}$ of $\Gamma_1(N)$ in $\Gamma_0(N)$ and a right transversal $\mathcal{T}_{\mathrm{SL}_2(\mathbb{Z})}$ of $\Gamma_1(N)$ in $\mathrm{SL}_2(\mathbb{Z})$. Using these we compute the set

$$\mathcal{U} = \{U(t, T^i) : t \in \mathrm{SL}_2(\mathbb{Z}), 1 \le i \le N\} \cup \{U(t, S^k) : t \in \mathrm{SL}_2(\mathbb{Z}), 0 \le k \le 2\}.$$

Finally, we compute the Dedekind sums $S_{\chi_1,\chi_2}(\mathcal{T}_{\Gamma_0})$ and $S_{\chi_1,\chi_2}(\mathcal{T}_{\mathrm{SL}_2(\mathbb{Z})})$ using Definition 1.

Main Computation

We write $\gamma_0 = \gamma_1 g$, where $\gamma_1 \in \Gamma_1(N)$ and $g \in \mathcal{T}_{\Gamma_0}$. Let $g \mapsto \overline{g}$ denote the right coset representative function uniquely described by $\mathcal{T}_{\mathrm{SL}_2(\mathbb{Z})}$ per Definition 3. By Lemma 1,

$$S_{\chi_1,\chi_2}(\gamma_0) = S_{\chi_1,\chi_2}(\gamma_1) + S_{\chi_1,\chi_2}(g).$$

Since $g \in \mathcal{T}_{\Gamma_0}$, $S_{\chi_1,\chi_2}(g)$ has been precomputed, so now we are only concerned about $S_{\chi_1,\chi_2}(\gamma_1)$. Using Lemma 2, we write

$$\gamma_1 = \pm T^{a_1}ST^{a_2}S\ldots T^{a_{k-1}}ST^{a_k}.$$

Using Theorem 1, we can rewrite

$$\tau(\gamma_1) = U(\overline{p_1}, T^{a_1})U(\overline{p_1T^{a_1}}, S)U(\overline{p_2}, T^{a_2})U(\overline{p_2T^{a_2}}, S)\cdots U(\overline{p_k}, T^{a_k})U(\overline{p_kT^{a_k}}, \pm I) = \gamma_1, \qquad (1)$$

where

$$p_k = T^{a_1}ST^{a_2}S\ldots T^{a_{k-1}}S.$$

Now we apply Lemma 3. For each exponent of T, we write $a_i = q_iN + r_i$ with $0 \le r_i < N$. Then

$$U(\overline{p_i}, T^{a_i}) = U^{q_i}(\overline{p_i}, T^N)U(\overline{p_i}, T^{r_i}). \qquad (2)$$

We apply the Dedekind sum to (2). By Lemma 1,

$$S_{\chi_1,\chi_2}\left(U(\overline{p_i}, T^{a_i})\right) = q_iS_{\chi_1,\chi_2}\left(U(\overline{p_i}, T^N)\right) + S_{\chi_1,\chi_2}\left(U(\overline{p_i}, T^{r_i})\right). \qquad (3)$$

Using (1) and (2) we can express γ_1 as a product of elements in $\mathcal{U}$. Applying Lemma 1 to $S_{\chi_1,\chi_2}(\tau(\gamma_1))$ and expanding via (3), we see that the Dedekind sum of each term is precomputed.

An Example

Fix $\Gamma_0(9)$. Let $\chi_1 = \chi_2$ be the primitive character modulo 3 with conductors $q_1 = q_2 = 3$. We want to compute $S_{\chi_1,\chi_2}(\gamma_0)$ where

$$\gamma_0 = \begin{pmatrix} 17 & 32 \\ 9 & 17 \end{pmatrix}.$$

We compute a right transversal of $\Gamma_1(9)$ in $\Gamma_0(9)$ as

$$\mathcal{T}_{\Gamma_0} = \left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 5 & 1 \\ 9 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 7 & 3 \\ 9 & 4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 9 & 5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 4 & 3 \\ 9 & 7 \end{smallmatrix}\right), \left(\begin{smallmatrix} 8 & 7 \\ 9 & 8 \end{smallmatrix}\right)\right\}.$$

We compute a right transversal of $\Gamma_1(9)$ in $\mathrm{SL}_2(\mathbb{Z})$

$$\mathcal{T}_{\mathrm{SL}_2(\mathbb{Z})} = \left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 5 & 1 \\ 9 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 7 & 3 \\ 9 & 4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 9 & 5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 4 & 3 \\ 9 & 7 \end{smallmatrix}\right), \cdots, \left(\begin{smallmatrix} 5 & 8 \\ 8 & 13 \end{smallmatrix}\right), \left(\begin{smallmatrix} 5 & 3 \\ 8 & 5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 7 & 13 \\ 8 & 15 \end{smallmatrix}\right), \left(\begin{smallmatrix} 7 & 6 \\ 8 & 7 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 \\ 8 & 17 \end{smallmatrix}\right)\right\}.$$

Now we compute the set

$$\mathcal{U} = \{U(t, T^i) : t \in \mathcal{T}_{\mathrm{SL}_2(\mathbb{Z})}, 1 \le i \le 9\} \cup \{U(t, S^k) : t \in \mathcal{T}_{\mathrm{SL}_2(\mathbb{Z})}, 0 \le k \le 2\}.$$

Using Definition 1, we compute the Dedekind sums $S_{\chi_1,\chi_2}(\mathcal{T}_{\Gamma_0})$ and $S_{\chi_1,\chi_2}(\mathcal{U})$. Note $\gamma_0 = \gamma_1 g$ where

$$\gamma_1 = \left(\begin{smallmatrix} -152 & 137 \\ -81 & 73 \end{smallmatrix}\right) \in \Gamma_1(9) \qquad \text{and} \qquad g = \left(\begin{smallmatrix} 8 & 7 \\ 9 & 8 \end{smallmatrix}\right) \in \mathcal{T}_{\Gamma_0}.$$

Since $g \in \mathcal{T}_{\Gamma_0}$, $S_{\chi_1,\chi_2}(g)$ has been precomputed; thus, we only need concern ourselves with computing $S_{\chi_1,\chi_2}(\gamma_1)$. By Lemma 2 we compute

$$\gamma_1 = -T^1ST^{-2}ST^{-2}ST^{-2}ST^{-2}ST^{-2}ST^{-2}ST^{-2}ST^{-11}ST^{-1}.$$

Now we apply Theorem 1 with all p_i written in matrix forms.

$$\begin{aligned}\tau(\gamma_1) =& U\left(\overline{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}, T^1\right)U\left(\overline{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)}, S\right)U\left(\overline{\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)}, T^{-2}\right)U\left(\overline{\left(\begin{smallmatrix} 1 & -3 \\ 1 & -2 \end{smallmatrix}\right)}, S\right)\cdots \\ &\cdots U\left(\overline{\left(\begin{smallmatrix} -15 & -13 \\ -8 & -7 \end{smallmatrix}\right)}, T^{-11}\right)U\left(\overline{\left(\begin{smallmatrix} -15 & 152 \\ -8 & 81 \end{smallmatrix}\right)}, S\right)U\left(\overline{\left(\begin{smallmatrix} 152 & 15 \\ 81 & 8 \end{smallmatrix}\right)}, T^{-1}\right)U\left(\overline{\left(\begin{smallmatrix} 152 & -137 \\ 81 & -73 \end{smallmatrix}\right)}, -I\right) = \gamma_1.\end{aligned}$$

Applying Lemma 3 to each term of the above product, we get the following computation.

$$\begin{aligned}
U\left(\overline{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}, T^1\right) &= U^0\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), T^9\right)U\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), T^1\right) & U\left(\overline{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)}, S\right) &= U\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), S\right) \\
U\left(\overline{\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)}, T^{-2}\right) &= U^{-1}\left(\left(\begin{smallmatrix} 1 & 8 \\ 1 & 9 \end{smallmatrix}\right), T^9\right)U\left(\left(\begin{smallmatrix} 1 & 8 \\ 1 & 9 \end{smallmatrix}\right), T^7\right) & U\left(\overline{\left(\begin{smallmatrix} 1 & -3 \\ 1 & -2 \end{smallmatrix}\right)}, S\right) &= U\left(\left(\begin{smallmatrix} 1 & 6 \\ 1 & 7 \end{smallmatrix}\right), S\right) \\
&\vdots & &\vdots \\
U\left(\overline{\left(\begin{smallmatrix} -15 & -13 \\ -8 & -7 \end{smallmatrix}\right)}, T^{-11}\right) &= U^{-2}\left(\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right), T^9\right)U\left(\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right), T^7\right) & U\left(\overline{\left(\begin{smallmatrix} -15 & 152 \\ -8 & 81 \end{smallmatrix}\right)}, S\right) &= U\left(\left(\begin{smallmatrix} 1 & 8 \\ 1 & 9 \end{smallmatrix}\right), S\right) \\
U\left(\overline{\left(\begin{smallmatrix} 152 & 15 \\ 81 & 8 \end{smallmatrix}\right)}, T^{-1}\right) &= U^{-1}\left(\left(\begin{smallmatrix} 8 & 7 \\ 9 & 8 \end{smallmatrix}\right), T^9\right)U\left(\left(\begin{smallmatrix} 8 & 7 \\ 9 & 8 \end{smallmatrix}\right), T^8\right) & U\left(\overline{\left(\begin{smallmatrix} 152 & -137 \\ 81 & -73 \end{smallmatrix}\right)}, -I\right) &= U\left(\left(\begin{smallmatrix} 8 & 7 \\ 9 & 8 \end{smallmatrix}\right), S^2\right)
\end{aligned}$$

Note that every term on the right hand side of these equalities are in the precomputed set $\mathcal{U}$. Thus, using Lemma 1 we know,

$$\begin{aligned}
S_{\chi_1,\chi_2}(\gamma_1) = &\ 0 \cdot S_{\chi_1,\chi_2}\left(U\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), T^9\right)\right) + S_{\chi_1,\chi_2}\left(U\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), T^1\right)\right) + S_{\chi_1,\chi_2}\left(U\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), S\right)\right) \\
&-1 \cdot S_{\chi_1,\chi_2}\left(U\left(\left(\begin{smallmatrix} 1 & 8 \\ 1 & 9 \end{smallmatrix}\right), T^9\right)\right) + S_{\chi_1,\chi_2}\left(U\left(\left(\begin{smallmatrix} 1 & 8 \\ 1 & 9 \end{smallmatrix}\right), T^7\right)\right) + S_{\chi_1,\chi_2}\left(U\left(\left(\begin{smallmatrix} 1 & 6 \\ 1 & 7 \end{smallmatrix}\right), S\right)\right) \\
&\vdots \\
&-2 \cdot S_{\chi_1,\chi_2}\left(U\left(\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right), T^9\right)\right) + S_{\chi_1,\chi_2}\left(U\left(\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right), T^7\right)\right) + S_{\chi_1,\chi_2}\left(U\left(\left(\begin{smallmatrix} 1 & 8 \\ 1 & 9 \end{smallmatrix}\right), S\right)\right) \\
&-1 \cdot S_{\chi_1,\chi_2}\left(U\left(\left(\begin{smallmatrix} 8 & 7 \\ 9 & 8 \end{smallmatrix}\right), T^9\right)\right) + S_{\chi_1,\chi_2}\left(U\left(\left(\begin{smallmatrix} 8 & 7 \\ 9 & 8 \end{smallmatrix}\right), T^8\right)\right) + S_{\chi_1,\chi_2}\left(U\left(\left(\begin{smallmatrix} 8 & 7 \\ 9 & 8 \end{smallmatrix}\right), S^2\right)\right).
\end{aligned}$$

Now, using the precomputed Dedekind sums, $S_{\chi_1,\chi_2}(\gamma_0) = S_{\chi_1,\chi_2}(\gamma_1) + S_{\chi_1,\chi_2}(g) = 0$.

Code

The final condensed code necessary to use and implement this algorithm can be found and cloned from this Github repository:
`https://github.com/prestontranbarger/NFDSFastComputation`

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